Math 152 - Calculus II - Test 4

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Ratio Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms, and suppose that

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = L.$$

Then there are three possibilities:

1. If $L < 1$, then the series converges.
2. If $L > 1$ or $L = \infty$, then the series diverges.
3. If $L = 1$, then NO INFO.

Note: this is a good test to try when there are terms like $n!$ or c^n, where c is a constant.

Root Test

Let $\sum_{n=1}^{\infty} a_n$ be a series with positive terms, and suppose that

$$\lim_{n\to\infty} (a_n)^{1/n} = L.$$

Then there are three possibilities:

1. If $L < 1$, then the series converges.
2. If $L > 1$ or $L = \infty$, then the series diverges.
3. If $L = 1$, then NO INFO.

Note: this is a good test to try when there are terms like n^n, $f(n)^n$, $f(n)^{cn}$ or $f(n)^{g(n)}$.

Alternating Series

An **alternating series** is series of the form

$$\sum_{n=0}^{\infty} (-1)^n a_n = a_1 - a_2 + a_3 - a_4 + \cdots ,$$

or

$$\sum_{n=0}^{\infty} (-1)^{n+1} a_n = -a_1 + a_2 - a_3 + a_4 - \cdots .$$

Alternating Series Test

For an alternating series (in either of the forms) if both

1. the sequence $\{a_n\}$ is decreasing, and
2. $\lim_{n\to\infty} a_n = 0$,

then the alternating series converges.

Note: the alternating series test says nothing about divergence. To determine if an alternating series diverges try the Test for Divergence, the Ratio Test for Absolute Convergence, or something else.

Absolute Convergence

If the series $\sum_{n=0}^{\infty} |a_n|$ converges, then the series $\sum_{n=0}^{\infty} a_n$ converges. Also, in this case we say that $\sum_{n=0}^{\infty} a_n$ **absolutely converges** or is **absolutely convergent**.

Note: absolute convergence is stronger! Absolute convergence $\Longrightarrow$ convergence.

Conditional Convergence

If the series $\sum_{n=0}^{\infty} a_n$ converges, but the series $\sum_{n=0}^{\infty} |a_n|$ diverges, then we say that $\sum_{n=0}^{\infty} a_n$ **conditionally converges** or is **conditionally convergent**.

Note: a good example of a conditionally convergent series to remember is the **alternating harmonic series**:

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots .$$

Ratio Test for Absolute Convergence

Let $\sum_{n=1}^{\infty} a_n$ be a series with nonzero terms, and suppose that

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = L.$$

Then there are three possibilities:

1. If $L < 1$, then the series converges absolutely (and therefore converges).
2. If $L > 1$ or $L = \infty$, then the series diverges.
3. If $L = 1$, then NO INFO.

Maclaurin Series

The n**th Maclaurin polynomial of** $f(x)$ is:

$$\sum_{k=0}^{n} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \cdots + \frac{f^{(n)}(0)}{n!} x^n.$$

The **Maclaurin series** of $f(x)$ is:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} x^k = f(0) + f'(0)x + \frac{f''(0)}{2!} x^2 + \cdots .$$

Taylor Series

The n**th Taylor polynomial of** $f(x)$ **centered at** x_0 is:

$$\sum_{k=0}^{n} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n.$$

The **Taylor series of** $f(x)$ **centered at** x_0 is:

$$\sum_{k=0}^{\infty} \frac{f^{(k)}(x_0)}{k!} (x-x_0)^k = f(x_0) + f'(x_0)(x-x_0) + \frac{f''(x_0)}{2!}(x-x_0)^2 + \cdots .$$

Note: a Maclaurin polynomial for $f(x)$ is a Taylor polynomial for $f(x)$ centered at 0 (i.e, $x_0 = 0$). Also, a Macluarin series for $f(x)$ is a Taylor series for $f(x)$ centered at 0 (i.e., $x_0 = 0$).

Power Series

A **power series centered at** x_0 (or a **power series in** $x - x_0$) is a series of the form

$$\sum_{n=0}^{\infty} c_n (x - x_0)^n = c_0 + c_1(x - x_0) + c_2(x - x_0)^2 + \cdots .$$

BIG QUESTION: for what values of x does the power series converge?

There are only three possibilities:

1. The power series only converges at $x = x_0$.
2. The power series converges (absolutely) for all real x.
3. There exists a positive real number $R > 0$, such that the power series converges (absolutely) for all $|x - x_0| < R$ (so $x_0 - R < x < x_0 + R$), the power series diverges for all $|x - x_0| > R$ (so $x < x_0 - R$ or $x_0 + R < x$) and the power series may converge or diverge when $|x - x_0| = R$ (so $x = x_0 - R$ or $x = x_0 + R$).

To test this you will always use the **RATIO TEST FOR ABSOLUTE CONVERGENCE**. In the event of case 3 above, you will need to test the endpoints and use some other test. That is, substitute the endpoint $x = x_0 - R$ into the power series and use some test other than the ratio test for absolute convergence (similarly, for $x = x_0 + R$).

The **interval of convergence** for a power series is the interval of all x values for which the power series converges. The center of the power series will be the center of the interval of convergence. The interval of convergence is $[x_0, x_0] = \{x_0\}$ (a single point) in case 1, $(-\infty, \infty)$ in case 2, and one of four possibilities in case 3 depending on whether the endpoints are included or not as mentioned above.

The **radius of convergence** is measure of how wide the interval of convergence is. The radius of convergence is $R = 0$ in case 1, $R = \infty$ in case 2, and the R mentioned in case 3.

Note 1: Macluarin and Taylor series are examples of power series.

Note 2: a power series will always converge at its center x_0.

Important Maclaurin Series

1. $\dfrac{1}{1-x} = \displaystyle\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots,$ where $-1 < x < 1$.
2. $e^x = \displaystyle\sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots,$ where $-\infty < x < \infty$.
3. $\sin(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots,$ where $-\infty < x < \infty$.
4. $\cos(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots,$ where $-\infty < x < \infty$.
5. $\dfrac{1}{1+x} = \displaystyle\sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots,$ where $-1 < x < 1$.
6. $\dfrac{1}{1+x^2} = \displaystyle\sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \cdots,$ where $-1 < x < 1$.
7. $\tan^{-1}(x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots,$ where $-1 \le x \le 1$.
8. $\ln(1+x) = \displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n x^{n+1}}{n+1} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots,$ where $-1 < x \le 1$.

Differentiating and Integrating Power Series

Let $f(x) = \sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x + c_2 x^2 + c_3 x^3 \cdots.$, for some interval of convergence with $R > 0$. Then we can differentiate and integrate the power series term by term (over the interior of the interval of convergence):

1. $f'(x) = c_1 + 2c_2 x + 3c_3 x^2 + \cdots = \displaystyle\sum_{n=0}^{\infty} n c_n x^{n-1}.$
2. $\displaystyle\int f(x)dx = C + c_0 x + \frac{c_1 x^2}{2} + \frac{c_3 x^3}{3} + \cdots = C + \sum_{n=0}^{\infty} \frac{c_n x^{n+1}}{n+1}.$

and the radius of convergence is R for both 1 and 2.

Note: you can do the same for general power series centered at x_0.