1. Evaluate $\int \cos^3(5x)\sin^5(5x)dx$.

SOLUTION 1:

$$\int \cos^3(5x)\sin^5(5x)dx = \int \cos(5x)\cos^2(5x)\sin^5(5x)\,dx$$
$$= \int \cos(5x)\left(1-\sin^2(5x)\right)\sin^5(5x)\,dx$$
$$u=\sin(5x) \Longrightarrow \frac{du}{dx} = 5\cos(5x)$$
$$= \int \cancel{\cos(5x)}(1-u^2)\,u^5\,\frac{du}{5\cancel{\cos(5x)}}$$
$$= \frac{1}{5}\int u^5 - u^7\;du$$
$$= \frac{1}{5}\cdot\frac{u^6}{6} - \frac{1}{5}\frac{u^8}{8} + C$$
$$= \boxed{\frac{\sin^6(5x)}{30} - \frac{\sin^8(5x)}{40} + C}$$

SOLUTION 2:

$$\int \cos^3(5x)\sin^5(5x)\,dx = \int \cos^3(5x)\sin^4(5x)\sin(5x)dx$$
$$= \int \cos^3(5x)\left(1-\cos^2(5x)\right)^2\sin(5x)\,dx$$
$$u=\cos(5x) \Longrightarrow \frac{du}{dx} = -5\sin(5x)$$
$$= \int u^3(1-u^2)^2\,\cancel{\sin(5x)}\,\frac{du}{-5\cancel{\sin(5x)}}$$
$$= -\frac{1}{5}\int u^3(1-u^2)^2\,du$$
$$= -\frac{1}{5}\int u^3(1-2u^2+u^4)\,du$$
$$= -\frac{1}{5}\int u^3-2u^5+u^7\;du$$
$$= -\frac{1}{5}\frac{u^4}{4} + \frac{2}{5}\frac{u^6}{6} - \frac{1}{5}\frac{u^8}{8} + C$$
$$= \boxed{-\frac{\cos^4(5x)}{20} + \frac{2\cos^6(5x)}{30} - \frac{\cos^8(5x)}{40} + C}$$

2. Evaluate $\int \cos^2(x)\sin^2(x)dx$.

SOLUTION :

$$\int \cos^2(x)\sin^2(x)dx = \int \tfrac{1}{2}\left(1+\cos(2x)\right)\cdot\tfrac{1}{2}\left(1-\cos(2x)\right)dx$$
$$= \frac{1}{4}\int 1 - \cos^2(2x)\;dx$$
$$= \frac{1}{4}\int 1 - \frac{1}{2}\left(1+\cos(4x)\right)dx$$
$$= \frac{1}{4}\int 1 - \frac{1}{2} - \frac{1}{2}\cos(4x)\;dx$$
$$= \frac{1}{4}\int \frac{1}{2} - \frac{1}{2}\cos(4x)\,dx$$
$$= \frac{1}{4}\cdot\frac{1}{2}x - \frac{1}{4}\cdot\frac{1}{2}\cdot\frac{\sin(4x)}{4} + C$$
$$= \boxed{\frac{x}{8} - \frac{\sin(4x)}{32} + C}$$

3. Evaluate $\int \frac{\sqrt{x^2-4}}{x}dx.$

$x = 2\sec\theta, \quad 0 \le \theta < \frac{\pi}{2} \text{ OR } \pi \le \theta < \frac{3\pi}{2}$

$\frac{dx}{d\theta} = 2\sec\theta\tan\theta \Rightarrow dx = 2\sec\theta\tan\theta\, d\theta$

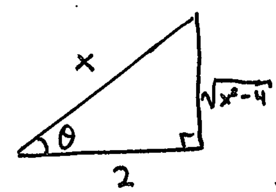
$\sqrt{x^2-4} = \sqrt{4\sec^2\theta - 4} = \sqrt{4(\sec^2\theta - 1)} = \sqrt{4}\sqrt{\tan^2\theta} = 2\tan\theta$

$$\int \frac{\sqrt{x^2-4}}{x}dx = \int \frac{\cancel{2}\tan\theta}{\cancel{2\sec\theta}}\, 2\cancel{\sec\theta}\tan\theta\, d\theta$$

$$= 2\int \tan^2\theta\, d\theta$$

$$= 2\int (\sec^2\theta - 1)\, d\theta$$

$$= 2\tan\theta - 2\theta + C$$



$\frac{x}{2} = \sec\theta$

$\Downarrow$

$\tan\theta = \frac{\sqrt{x^2-4}}{2}$

$$= \boxed{2\frac{\sqrt{x^2-4}}{2} - 2\sec^{-1}\left(\frac{x}{2}\right) + C}$$

$$= \boxed{\sqrt{x^2-4} - 2\sec^{-1}\left(\frac{x}{2}\right) + C}$$

4. Evaluate $\int \frac{x-1}{x^2+3x+2}dx.$

Step 1: deg of numerator is 1, deg of denominator is 2 } No long division is needed.

Step 2:

$$\frac{x-1}{x^2+3x+2} = \frac{x-1}{(x+2)(x+1)} = \frac{A}{x+2} + \frac{B}{x+1}$$

$$= \frac{A(x+1)}{(x+2)(x+1)} + \frac{B(x+2)}{(x+1)(x+2)}$$

$$= \frac{A(x+1) + B(x+2)}{(x+2)(x+1)}$$

so $x - 1 = A(x+1) + B(x+2)$

$x - 1 = Ax + A + Bx + 2B$

$x - 1 = (A+B)x + A + 2B$

Hence, $1 = A + B$ and $-1 = A + 2B$

$A = 1 - B \longrightarrow -1 = (1-B) + 2B$

$-1 = 1 + B$

$A = 1 - (-2) \longleftarrow B = -2$

$A = 3$

$$\int \frac{x-1}{x^2+3x+2}dx = \int \left[\frac{3}{x+2} + \frac{-2}{x+1}\right]dx = \boxed{3\ln|x+2| - 2\ln|x+1| + C}$$

5. Evaluate $\int_{-\infty}^{0} e^{5x}\,dx$.

$$\int_{-\infty}^{0} e^{5x}\,dx = \lim_{t\to-\infty} \int_t^0 e^{5x}\,dx$$

$$= \lim_{t\to-\infty} \left[\frac{1}{5}e^{5x}\right]_t^0$$

$$= \lim_{t\to-\infty} \left[\frac{1}{5}e^{5\cdot 0} - \frac{1}{5}e^{5t}\right]$$

$$= \lim_{t\to-\infty} \left[\frac{1}{5}(1) - \frac{1}{5}e^{5t}\right]$$

$$= \lim_{t\to-\infty} \left[\frac{1}{5} - \frac{1}{5}e^{5t}\right]$$

$$= \frac{1}{5} - 0$$

$$= \boxed{\frac{1}{5}} \quad \underline{\text{convergent}}$$

6. Evaluate $\int_4^5 \frac{1}{x-5}\,dx$.

$\frac{1}{x-5}$ is discontinuous at $x=5$. Thus, the integral is improper:

$$\int_4^5 \frac{1}{x-5}\,dx = \lim_{t\to5^-} \int_4^t \frac{1}{x-5}\,dx$$

$$= \lim_{t\to5^-} \Big[\ln|x-5|\Big]_4^t$$

$$= \lim_{t\to5^-} \Big[\ln|t-5| - \ln|4-5|\Big]$$

$$= \lim_{t\to5^-} \Big[\ln|t-5| - \ln|-1|\Big]$$

$$= \lim_{t\to5^-} \Big[\ln|t-5| - \underbrace{\ln(1)}_{=0}\Big]$$

$$= -\infty - 0$$

$$= \boxed{-\infty} \quad \underline{\text{Divergent}}$$

7. (a) Perform long division on the following rational function to find the missing constants:

$$\frac{x^3-1}{x+2} = ax^2 + bx + c + \frac{d}{x+2}.$$

$$\begin{array}{r} x^2 - 2x + 4 \\ x+2 \overline{) x^3 \qquad\qquad -1} \\ -(x^3+2x^2) \\ \hline -2x^2 \\ -(-2x^2-4x) \\ \hline 4x - 1 \\ -(4x+8) \\ \hline -9 \end{array}$$

So,

$$\boxed{\frac{x^3-1}{x+2} = x^2 - 2x + 4 + \frac{-9}{x+2}}$$

OR

$$\boxed{\begin{array}{l} a = 1 \\ b = -2 \\ c = 4 \\ d = -9 \end{array}}$$

(b) Use part (a) to evaluate $\int \frac{x^3-1}{x+2} dx$.

$$\int \frac{x^3-1}{x+2} dx = \int \left(x^2 - 2x + 4 - \frac{9}{x+2}\right) dx$$

$$= \frac{x^3}{3} - \frac{2x^2}{2} + 4x - 9\ln|x+2| + C$$

$$= \boxed{\frac{x^3}{3} - x^2 + 4x - 9\ln|x+2| + C}$$

8. Write out the FORM of the partial fraction decomposition for the following (DO NOT find the numerical values for the unknown coefficients).

(a) $\frac{x^3+x^2+1}{x(x-1)(x^2+1)^2} = \boxed{\frac{A}{x} + \frac{B}{x-1} + \frac{Cx+D}{x^2+1} + \frac{Ex+F}{(x^2+1)^2}}$

(b) $\frac{x^2+10}{x^3(x^2+4)} = \boxed{\frac{A}{x} + \frac{B}{x^2} + \frac{C}{x^3} + \frac{Dx+E}{x^2+4}}$

(c) $\frac{4x-1}{(x-4)^2(x+3)(x^2+9)} = \boxed{\frac{A}{x-4} + \frac{B}{(x-4)^2} + \frac{C}{x+3} + \frac{Dx+E}{x^2+9}}$