POLYNOMIAL LONG DIVISION & INTEGRATION

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Example 0.1. Evaluate:

$$\int \frac{x^3 - 5}{x + 1} dx.$$

Notice that the degree of the numerator is 3 and the degree of the denominator is 1 so we must perform polynomial long division described above.

$$
\begin{array}{r}
x^2 - x + 1 \\
x+1 \overline{\smash{)}\, x^3 \qquad\qquad - 5} \\
\underline{- x^3 - x^2 \qquad\quad} \\
- x^2 \qquad\quad \\
\underline{x^2 + x \quad} \\
x - 5 \\
\underline{- x - 1} \\
- 6
\end{array}
$$

Therefore,

$$
\begin{aligned}
\int \frac{x^3 - 5}{x + 1} dx &= \int \left(x^2 - x + 1 - \frac{6}{x + 1} \right) dx \\
&= \frac{1}{3}x^3 - \frac{1}{2}x^2 + x - 6 \ln|x + 1| + C.
\end{aligned}
$$

Example 0.2. Evaluate:

$$\int \frac{x^4 - 1}{x^2 + 1} dx.$$

Notice that the degree of the numerator is 4 and the degree of the denominator is 2 so we must perform polynomial long division, or do we!?

SOLUTION 1: Notice that we can factor and reduce the problem:

$$\int \frac{x^4 - 1}{x^2 + 1} dx = \int \frac{\cancel{(x^2 + 1)}(x^2 - 1)}{\cancel{x^2 + 1}} dx = \int (x^2 - 1) dx = \frac{1}{3}x^3 - x + C.$$

SOLUTION 2: We can still use polynomial long division to solve the problem:

$$
\begin{array}{r}
x^2 - 1 \\
x^2+1 \overline{\smash{\big)}\; x^4 \qquad\quad - 1} \\
\underline{- x^4 - x^2 \qquad} \\
- x^2 - 1 \\
\underline{x^2 + 1} \\
0
\end{array}
$$

Therefore,

$$
\begin{aligned}
\int \frac{x^4-1}{x^2+1} dx &= \int \left(x^2 - 1 + \frac{0}{x^2+1} \right) dx \\
&= \int (x^2 - 1) dx \\
&= \frac{1}{3} x^3 - x + C.
\end{aligned}
$$

Example 0.3. Perform polynomial long division on:

$$\frac{x^2-5}{x-4}.$$

SOLUTION:

$$
\begin{array}{r}
x \quad + 4 \\
x-4 \overline{\smash{\big)}\; x^2 \qquad\quad - 5} \\
\underline{- x^2 + 4x \qquad} \\
4x \quad - 5 \\
\underline{- 4x + 16} \\
11
\end{array}
$$

Therefore,

$$\frac{x^2-5}{x-4} = x + 4 + \frac{11}{x-4}.$$

Example 0.4. Perform polynomial long division on:

$$\frac{x^2+4}{x+2}.$$

SOLUTION:

$$
\begin{array}{r}
x - 2 \\
x+2 \overline{\smash{\big)}\; x^2 \qquad\quad + 4} \\
\underline{- x^2 - 2x \qquad} \\
- 2x + 4 \\
\underline{2x + 4} \\
8
\end{array}
$$

Therefore,

$$\frac{x^2+4}{x+2} = x - 2 + \frac{8}{x+2}.$$

Example 0.5. Perform polynomial long division on:

$$\frac{x^6 - 2x^3 + 1}{x + 2}.$$

SOLUTION:

$$
\begin{array}{r|rrrrrrr}
 & x^5 & -2x^4 & +4x^3 & -10x^2 & +20x & -40 & \\
\hline
x+2 & x^6 & & & -2x^3 & & & +1 \\
 & -x^6 & -2x^5 & & & & & \\
\cline{2-3}
 & & -2x^5 & & & & & \\
 & & 2x^5 & +4x^4 & & & & \\
\cline{3-4}
 & & & 4x^4 & -2x^3 & & & \\
 & & & -4x^4 & -8x^3 & & & \\
\cline{4-5}
 & & & & -10x^3 & & & \\
 & & & & 10x^3 & +20x^2 & & \\
\cline{5-6}
 & & & & & 20x^2 & & \\
 & & & & & -20x^2 & -40x & \\
\cline{6-7}
 & & & & & & -40x & +1 \\
 & & & & & & 40x & +80 \\
\cline{7-8}
 & & & & & & & 81
\end{array}
$$

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