Math 152 - Fall 2012 - Quiz #5 **Score**: ________ out of 10.

Name: Key

Show all work clearly and in order. Please box your answers.

1. Evaluate $\displaystyle\int \frac{x^2}{\sqrt{9-x^2}}dx.$

Let $x = 3\sin\theta$ with $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$

$$\frac{dx}{d\theta} = 3\cos\theta \Rightarrow dx = 3\cos\theta\, d\theta$$

Recall $\sin^2\theta + \cos^2\theta = 1$ since $\frac{-\pi}{2} \le \theta \le \frac{\pi}{2}$

$$\sqrt{9-x^2} = \sqrt{9-(3\sin\theta)^2} = \sqrt{9-9\sin^2\theta} = \sqrt{9(1-\sin^2\theta)} = \sqrt{9\cos^2\theta} = 3\cos\theta$$

so

Recall $\sin^2\theta = \frac{1}{2}(1-\cos(2\theta))$

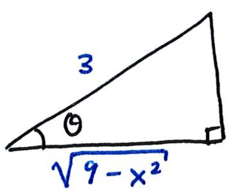
$$\int \frac{x^2}{\sqrt{9-x^2}}dx = \int \frac{(3\sin\theta)^2}{\cancel{3\cos\theta}} \cdot \cancel{3\cos\theta}\, d\theta = \int 9\sin^2\theta\, d\theta = \int 9\cdot\frac{1}{2}(1-\cos(2\theta))d\theta =$$

Recall $\sin(2\theta) = 2\sin\theta\cos\theta$

$$= \frac{9}{2}\left[\theta - \frac{\sin(2\theta)}{2}\right] + C = \frac{9}{2}\left[\theta - \frac{2\sin\theta\cos\theta}{2}\right] + C$$

$x = 3\sin\theta$

$\frac{x}{3} = \sin\theta$



so $\theta = \sin^{-1}\left(\frac{x}{3}\right)$

$\sin\theta = \frac{x}{3}$

$\cos\theta = \frac{\sqrt{9-x^2}}{3}$

$$= \frac{9}{2}\left[\sin^{-1}\left(\frac{x}{3}\right) - \frac{x}{3}\,\frac{\sqrt{9-x^2}}{3}\right] + C$$

$$= \boxed{\frac{9}{2}\sin^{-1}\left(\frac{x}{3}\right) - \frac{x\sqrt{9-x^2}}{2} + C}$$

2. Evaluate $\displaystyle\int \frac{1}{x^2\sqrt{1+4x^2}}dx.$

Notice that
$$\int \frac{1}{x^2\sqrt{1+4x^2}}dx = \int \frac{1}{x^2\sqrt{4(\frac{1}{4}+x^2)}}dx = \int \frac{1}{x^2\cdot 2\cdot\sqrt{\frac{1}{4}+x^2}}dx = \frac{1}{2}\int \frac{1}{x^2\sqrt{\frac{1}{4}+x^2}}dx$$

Let $x = \frac{1}{2}\tan\theta$ with $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$

$$\frac{dx}{d\theta} = \frac{1}{2}\sec^2\theta \Rightarrow dx = \frac{1}{2}\sec^2\theta\, d\theta$$

Recall $\tan^2\theta + 1 = \sec^2\theta$ since $\frac{-\pi}{2} < \theta < \frac{\pi}{2}$

$$\sqrt{\tfrac{1}{4}+x^2} = \sqrt{\tfrac{1}{4}+\left(\tfrac{1}{2}\tan\theta\right)^2} = \sqrt{\tfrac{1}{4}+\tfrac{1}{4}\tan^2\theta} = \sqrt{\tfrac{1}{4}(1+\tan^2\theta)} = \sqrt{\tfrac{1}{4}\sec^2\theta} = \tfrac{1}{2}\sec\theta$$

so
$$\frac{1}{2}\int\frac{1}{x^2\sqrt{\frac{1}{4}+x^2}}dx = \frac{1}{2}\int\frac{1}{(\frac{1}{2}\tan\theta)^2(\frac{1}{2}\sec\theta)}\cdot\frac{1}{2}\sec^2\theta\, d\theta = \frac{1}{2}\int\frac{1}{\frac{1}{4}\tan^2\theta\,\frac{1}{2}\sec\theta}\cdot\frac{1}{2}\sec^2\theta\, d\theta =$$

$$= 2\int\frac{\sec\theta}{\tan^2\theta}d\theta = 2\int\frac{\left(\frac{1}{\cos\theta}\right)}{\left(\frac{\sin\theta}{\cos\theta}\right)^2}d\theta = 2\int\frac{\cos\theta}{\sin^2\theta}d\theta$$

Let $t = \sin\theta \Rightarrow \frac{dt}{d\theta} = \cos\theta \Rightarrow d\theta = \frac{dt}{\cos\theta}$

$$= 2\int\frac{\cancel{\cos\theta}}{t^2}\cdot\frac{dt}{\cancel{\cos\theta}} = 2\int t^{-2}\,dt =$$

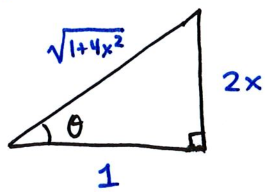
$$= 2\left[\frac{t^{-1}}{-1}\right] + C$$

$$= \frac{-2}{t} + C$$

$$= \frac{-2}{\sin\theta} + C$$

$x = \frac{1}{2}\tan\theta$

$2x = \tan\theta$



so $\sin\theta = \dfrac{2x}{\sqrt{1+4x^2}}$

$$= \frac{-2}{\left(\frac{2x}{\sqrt{1+4x^2}}\right)} + C$$

$$= \boxed{-\frac{\sqrt{1+4x^2}}{x} + C}$$