TEST 3

Math 152 - Calculus II

Score: ________ out of 100

Name: Key

Read all of the following information before starting the exam:

- You have 50 minutes to complete the exam.
- Show all work, clearly and in order, if you want to get full credit. Please make sure you read the directions for each problem. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Please box/circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements.
- This test has 7 problems and is worth 100 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

1. Determine whether the sequence converges, and if so find its limit.

(a) $\left\{ \dfrac{4n^3 - 2n + 1}{3n^3 + 2n^2 - 4} \right\}_{n=1}^{\infty}$

SOL 1: $\lim_{n\to\infty} \dfrac{4n^3-2n+1}{3n^3+2n^2-4} = \lim_{n\to\infty} \dfrac{(4n^3-2n+1)\left(\frac{1}{n^3}\right)}{(3n^3+2n^2-4)\left(\frac{1}{n^3}\right)} = \lim_{n\to\infty} \dfrac{4-\frac{2}{n^2}+\frac{1}{n^3}}{3+\frac{2}{n}-\frac{4}{n^3}} = \boxed{\frac{4}{3}}$

converges

SOL 2: Let $f(x) = \dfrac{4x^3-2x+1}{3x^3+2x^2-4}$ Type $\frac{\infty}{\infty}$

$\lim_{x\to\infty} \dfrac{4x^3-2x+1}{3x^3+2x^2-4} \overset{L'H}{=} \lim_{x\to\infty} \dfrac{12x^2-2}{9x^2+4x} \overset{L'H}{=} \lim_{x\to\infty} \dfrac{24x}{18x+4} \overset{L'H}{=} \lim_{x\to\infty} \dfrac{24}{18} = \dfrac{24}{18} = \boxed{\frac{4}{3}}$

So $\lim_{n\to\infty} a_n = \frac{4}{3}$, converges

(b) $\left\{ \dfrac{(\ln n)^2}{3n} \right\}_{n=1}^{\infty}$

Let $f(x) = \dfrac{(\ln(x))^2}{3x}$

Type $\frac{\infty}{\infty}$ Type $\frac{\infty}{\infty}$

$\lim_{x\to\infty} f(x) = \lim_{x\to\infty} \dfrac{(\ln(x))^2}{3x} \overset{L'H}{=} \lim_{x\to\infty} \dfrac{2\ln(x)\cdot\left(\frac{1}{x}\right)}{3} = \lim_{x\to\infty} \dfrac{2\ln(x)}{3x} \overset{L'H}{=} \lim_{x\to\infty} \dfrac{2\left(\frac{1}{x}\right)}{3} = \boxed{0}$

So $\lim_{n\to\infty} \dfrac{(\ln(n))^2}{3n} = 0$, converges

(c) $\left\{ \dfrac{\cos(n)}{n^2} \right\}_{n=1}^{\infty}$ (*Hint*: use the Squeeze Theorem)

Notice that $\dfrac{-1}{n^2} \leq \dfrac{\cos(n)}{n^2} \leq \dfrac{1}{n^2}$,

and $\lim_{n\to\infty} -\dfrac{1}{n^2} = 0$

and $\lim_{n\to\infty} \dfrac{1}{n^2} = 0$

Hence, by the Squeeze Thm. $\lim_{n\to\infty} \dfrac{\cos(n)}{n^2} = \boxed{0}$

converges

2. Show that the given sequence is strictly increasing or strictly decreasing.

$$\left\{ \frac{5n}{3n+1} \right\}_{n=1}^{\infty}$$

SOL 1: $a_{n+1} - a_n$

$= \dfrac{5(n+1)}{3(n+1)+1} - \dfrac{5n}{3n+1}$

$= \dfrac{5n+5}{3n+4} - \dfrac{5n}{3n+1}$

$= \dfrac{(5n+5)(3n+1) - 5n(3n+4)}{(3n+4)(3n+1)}$

$= \dfrac{\cancel{15n^2} + \cancel{20n} + 5 - \cancel{15n^2} - \cancel{20n}}{(3n+4)(3n+1)}$

$= \dfrac{5}{(3n+4)(3n+1)} > 0$ ⊕

⊕ ⊕ Hence.

SOL 2: $\dfrac{a_{n+1}}{a_n} = \dfrac{\frac{5(n+1)}{3(n+1)+1}}{\frac{5n}{3n+1}}$

$= \left(\dfrac{5n+5}{3n+4}\right)\cdot\left(\dfrac{3n+1}{5n}\right)$

$= \dfrac{15n^2+20n+5}{15n^2+20n} > 1$

numerator is always 5 more than the denominator

Hence, →

SOL 3: Let $f(x) = \dfrac{5x}{3x+1}$

$f'(x) = \dfrac{(3x+1)(5) - 5x(3)}{(3x+1)^2}$

$= \dfrac{15x+5-15x}{(3x+1)^2}$

$= \dfrac{5}{(3x+1)^2} > 0$ ⊕ ⊕

So $f(x)$ is strictly increasing

Hence,

The sequence is strictly $\boxed{\text{increasing}}$

3. Each series below is geometric. Determine both a and r. Then decide whether the series converges or diverges. If the series converges, then find its sum. If it diverges, write "NO SUM."

(a) $\displaystyle\sum_{k=1}^{\infty}\left(\frac{1}{\ln 2}\right)^{k-1}$

$a = \boxed{1}$

$r = \boxed{\frac{1}{\ln(2)}}$ since $\frac{1}{\ln(2)} \geq 1$ the geometric series diverges so,

sum= $\boxed{\text{NO SUM}}$

(b) $\displaystyle\sum_{k=1}^{\infty}(-1)^{k-1}\frac{2^{3k}}{9^{k+1}} = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{8^k}{9^2 9^{k-1}} = \sum_{k=1}^{\infty}(-1)^{k-1}\frac{8 \cdot 8^{k-1}}{9^2\, 9^{k-1}}$

$$= \sum_{k=1}^{\infty}\underbrace{\left(\frac{8}{9^2}\right)}_{a}\underbrace{\left(\frac{-8}{9}\right)}_{r}^{k-1}$$

$a = \boxed{\frac{8}{9^2} = \frac{8}{81}}$

$r = \boxed{\frac{-8}{9}}$

sum= $\boxed{\frac{8/81}{1+8/9} = \frac{8}{153}}$

4. Use the Divergence Test on each of the following to determine whether the given series diverges. If the test yields no conclusion, then be sure to say so. You must set up, evaluate, and interpret the correct limit to earn credit.

(a) $\displaystyle\sum_{n=1}^{\infty}\cos\left(\frac{2}{n}\right)$

$$\lim_{n\to\infty}\cos\left(\frac{2}{n}\right) = \cos(0) = 1 \neq 0 \quad \boxed{\text{diverges}}$$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{n^5+3}{3n^6-n^3+1}$

$$\lim_{n\to\infty}\frac{n^5+3}{3n^6-n^3+1} = \lim_{n\to\infty}\frac{(n^5+3)\left(\frac{1}{n^6}\right)}{(3n^6-n^3+1)\left(\frac{1}{n^6}\right)} = \lim_{n\to\infty}\frac{\frac{1}{n}+\frac{3}{n^6}}{3-\frac{1}{n^3}+\frac{1}{n^6}} = \frac{0+0}{3-0+0}$$

$$= 0$$

$\boxed{\text{NO INFO}}$

5. Use the Integral Test to determine whether the given series converges or diverges. Clearly identify the function $f(x)$ you are embedding the sequence of terms into. You may assume that $f(x)$ is positive, decreasing and continuous for $x \geq 1$, so you do not need to verify this. Just use the integral test and state your conclusion.

$$\sum_{n=2}^{\infty} \frac{\ln n}{n}$$

Let $f(x) = \dfrac{\ln(x)}{x}$

$$\int_2^{\infty} \frac{\ln(x)}{x}\,dx = \lim_{t\to\infty} \int_2^t \frac{\ln(x)}{x}\,dx$$

$$u = \ln(x) \Rightarrow \begin{array}{l} u(2) = \ln(2) \\ u(t) = \ln(t) \end{array}$$

$$\frac{du}{dx} = \frac{1}{x} \Rightarrow dx = x\,du$$

$$= \lim_{t\to\infty} \int_{\ln(2)}^{\ln(t)} \frac{u}{x}\cdot x\,du = \lim_{t\to\infty} \int_{\ln(2)}^{\ln(t)} u\,du = \lim_{t\to\infty} \left[\frac{u^2}{2}\right]_{\ln(2)}^{\ln(t)}$$

$$= \lim_{t\to\infty} \left[\frac{(\ln(t))^2}{2} - \frac{(\ln(2))^2}{2}\right]$$

$= \infty$ diverges so

by the Integral Test, the series $\boxed{\text{diverges}}$.

6. Use the Limit Comparison Test to determine whether the given series converges or diverges. Clearly write down what a_n and b_n are, compute the appropriate limit, determine the convergence or divergence of your comparison series $\sum_{n=1}^{\infty} b_n$, and then write your conclusion.

$$\sum_{n=1}^{\infty} \frac{5n}{n^4+1}$$

$$a_n = \frac{5n}{n^4+1} \qquad b_n = \frac{5n}{n^4} = \frac{5}{n^3}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{5n}{n^4+1}\right)}{\left(\frac{5}{n^3}\right)} = \lim_{n\to\infty} \left(\frac{5n}{n^4+1}\right)\cdot\left(\frac{n^3}{5}\right) = \lim_{n\to\infty} \frac{5n^4}{5n^4+5}$$

$$= \lim_{n\to\infty} \frac{(5n^4)\left(\frac{1}{n^4}\right)}{(5n^4+5)\left(\frac{1}{n^4}\right)}$$

$$= \lim_{n\to\infty} \frac{5}{5+\frac{5}{n^4}} = \frac{5}{5+0}$$

$$= \frac{5}{5} = 1$$

positive and finite!

Also, since $\displaystyle\sum_{n=1}^{\infty} \frac{5}{n^3} = 5 \underbrace{\sum_{n=1}^{\infty} \frac{1}{n^3}}_{\text{p-series}}$ $p = 3 > 1$ converges. By Limit Comparison Test $\displaystyle\sum_{n=1}^{\infty} \frac{5n}{n^4+1}$ $\boxed{\text{converges}}$

7. Determine whether the following series converges or diverges. (*Hint*: there are many ways to do this problem!)

$$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$$

$\boxed{\text{3 Methods!}}$

Comparison Test:

$$\frac{1}{n^2+1} \leq \frac{1}{n^2}$$

since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (it is a p-series with $p=2$)

$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ also $\boxed{\text{converges}}$ by the comparison test ✓

limit comparison test:

$$a_n = \frac{1}{n^2+1} \qquad b_n = \frac{1}{n^2}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{1}{n^2+1}\right)}{\left(\frac{1}{n^2}\right)} = \lim_{n\to\infty}\left(\frac{n^2}{n^2+1}\right)\frac{\left(\frac{1}{n^2}\right)}{\left(\frac{1}{n^2}\right)}$$

$$= \lim_{n\to\infty} \frac{1}{1+1/n^2} = \frac{1}{1+0} = 1$$

finite and positive!
So Both series converge OR
Both series diverge

since $\sum_{n=1}^{\infty} \frac{1}{n^2}$ converges (p-series with $p=2>1$)

$\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ also $\boxed{\text{converges}}$ by the limit comparison test ✓

Integral Test: consider $f(x) = \frac{1}{x^2+1}$ for $x \geq 1$. This function is continuous (rational with no real solution to $x^2+1=0$)

positive $\left(\frac{1 \leftarrow \oplus}{x^2+1 \leftarrow \oplus} > 0\right)$

decreasing $\left(f'(x) = \frac{-2x \leftarrow \ominus}{(x^2+1)^2 \leftarrow \oplus} < 0\right)$

$$\int_1^{\infty} \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^t \frac{1}{x^2+1}\,dx = \lim_{t\to\infty} \left[\tan^{-1}(x)\right]_1^t = \lim_{t\to\infty} \left[\tan^{-1}(t) - \tan^{-1}(1)\right] = \frac{\pi}{2} - \frac{\pi}{4} = \frac{\pi}{4}$$

converges

Hence, $\sum_{n=1}^{\infty} \frac{1}{n^2+1}$ also $\boxed{\text{converges}}$ by the Integral Test. ✓