# Math 152 - Calculus II - Test 3


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## Sequences

A **sequence** is a list of numbers:

$$\{a_n\}_{n=1}^{\infty} = \{a_1, a_2, a_3, \ldots\}.$$

The sequence $\{a_n\}_{n=1}^{\infty}$ **converges** (**convergent**) to $L$ if

$$\lim_{n\to\infty} a_n = L,$$

and **diverges** (**divergent**) otherwise.

## Embedding a Sequence

Consider a sequence $\{a_n\}_{n=1}^{\infty}$. Suppose $f(x)$ is a function of a real variable $x$ where $f(n) = a_n$ for $n = 1, 2, 3, \ldots$.
If

$$\lim_{x\to\infty} f(x) = L,$$

then

$$\lim_{n\to\infty} a_n = L.$$

This method allows us to use L'Hôpital's rule.

## Monotone Sequences

Consider a sequence $\{a_n\}_{n=1}^{\infty}$. The sequence is

**increasing** if $a_1 \leq a_2 \leq \cdots$, **3 WAYS TO CHECK (Use #1 if possible)**:

1. embedding the sequence into $f(x)$, which is increasing (i.e. $f'(x) \geq 0$), or
2. $a_{n+1} - a_n \geq 0$, or
3. $\frac{a_{n+1}}{a_n} \geq 1$.

**strictly increasing** if $a_1 < a_2 < \cdots$, **3 WAYS TO CHECK (Use #1 if possible)**:

1. embedding the sequence into $f(x)$, which is strictly increasing (i.e. $f'(x) > 0$), or
2. $a_{n+1} - a_n > 0$, or
3. $\frac{a_{n+1}}{a_n} > 1$.

**decreasing** if $a_1 \geq a_2 \geq \cdots$, **3 WAYS TO CHECK (Use #1 if possible)**:

1. embedding the sequence into $f(x)$, which is decreasing (i.e. $f'(x) \leq 0$), or
2. $a_{n+1} - a_n \leq 0$, or
3. $\frac{a_{n+1}}{a_n} \leq 1$.

**strictly decreasing** if $a_1 > a_2 > \cdots$, **3 WAYS TO CHECK (Use #1 if possible)**:

1. embedding the sequence into $f(x)$, which is strictly decreasing (i.e. $f'(x) < 0$), or
2. $a_{n+1} - a_n < 0$, or
3. $\frac{a_{n+1}}{a_n} < 1$.

## Squeeze (Sandwich) Theorem

Suppose $\{a_n\}_{n=1}^{\infty}$, $\{b_n\}_{n=1}^{\infty}$ and $\{c_n\}_{n=1}^{\infty}$ be sequences where with

$$\lim_{n\to\infty} a_n = L = \lim_{n\to\infty} b_n,$$

and

$$a_n \leq b_n \leq c_n \text{ (for all } n),$$

then $\{b_n\}_{n=1}^{\infty}$ also converges to $L$. That is,

$$\lim_{n\to\infty} b_n = L$$

## Series

A **series** (**infinite series**) is the sum of the terms of a sequence. That is, something in the form

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$$

This sum may be finite (**converge**) or not (**diverge**).
Every series has two sequences associated to it:

### The sequence of terms:

The sequence $\{a_k\}_{k=1}^{\infty}$

### The sequence of partial sums:

The sequence $\{s_n\}_{n=1}^{\infty}$, where $s_n$ is the $n$**th partial sum** defined as

$$s_n = \sum_{k=1}^{n} a_k = a_1 + a_2 + \cdots + a_n.$$

## Formal Definition of Convergence for Series:

The series

$$\sum_{k=1}^{\infty} a_k = a_1 + a_2 + a_3 + \cdots$$

**converges** to $S$ (has finite sum S) if sequence of partial sums converges to $S$:

$$\lim_{n\to\infty} s_n = S.$$

That is,

$$\lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \sum_{k=1}^{\infty} a_k = S.$$

Otherwise, the series **diverges**.

## Geometric Series

A series of the form

$$\sum_{n=0}^{\infty} ar^n = \sum_{n=1}^{\infty} ar^{n-1} = a + ar + ar^2 + \cdots,$$

is called a **geometric series**. Note: $a=$ first term, and $r=$ ratio of consecutive terms.
If $|r| < 1$, then the geometric series converges, and

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}.$$

If $|r| \geq 1$, then the geometric series diverges.

## $p$-Series

A series of the form

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = 1 + \frac{1}{2^p} + \frac{1}{3^p} + \cdots,$$

is called a $p$**-series**.

If $p > 1$, then the $p$-series converges.

If $p \leq 1$, then the $p$-series diverges.

### Harmonic Series

This is a $p$-series with $p = 1$:

$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \cdots,$$

which diverges.

## Test for Divergence

(i) If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges.

(ii) If $\lim_{n\to\infty} a_n = 0$, then NO INFO (the series may converge or diverge).

## Integral Test

Consider the series $\sum_{n=1}^{\infty} a_n$. If $f(x)$ is a function of a real variable $x$ with $f(n) = a_n$ for all $n = 1, 2, 3, \ldots$ that is continuous, positive and decreasing on $[1, \infty)$, then we can try to use the **Integral Test**:

$$\int_1^{\infty} f(x)dx \quad \text{AND} \quad \sum_{n=1}^{\infty} a_n$$

both converge OR both diverge.

That is,

(i) If $\int_1^{\infty} f(x)dx$ converges, then $\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $\int_1^{\infty} f(x)dx$ diverges, then $\sum_{n=1}^{\infty} a_n$ diverges.

## Comparison Test

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both series with positive terms.

(i) If $\sum_{n=1}^{\infty} b_n$ converges AND $a_n \leq b_n$ for all $n$, then $\sum_{n=1}^{\infty} a_n$ converges.

(ii) If $\sum_{n=1}^{\infty} b_n$ diverges AND $a_n \geq b_n$ for all $n$, then $\sum_{n=1}^{\infty} a_n$ diverges.

## Limit Comparison Test

Suppose $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=1}^{\infty} b_n$ are both series with positive terms.

If

$$\lim_{n\to\infty} \frac{a_n}{b_n} = c,$$

where $c$ is finite, and $c > 0$, then

$$\sum_{n=1}^{\infty} a_n \text{ AND } \sum_{n=1}^{\infty} b_n$$

both converge OR both diverge.