TEST 4

Math 152 - Calculus II **Score:** ________ out of 100

Name: ______________________________

Read all of the following information before starting the exam:

- You have 50 minutes to complete the exam.
- Show all work, clearly and in order, if you want to get full credit. Please make sure you read the directions for each problem. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Please box/circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements.
- This test has 6 problems and is worth 100 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

1. Determine if the following series converge or diverge. Clearly state the test you are using to obtain your answer.

(a) $\sum_{n=0}^{\infty} \frac{3^n}{n!}$.

Ratio Test: $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{3^{n+1}}{(n+1)!} \cdot \frac{n!}{3^n} = \lim_{n\to\infty} \frac{\cancel{3^n} \cdot 3 \cdot \cancel{n!}}{(n+1)\cancel{n!}\,\cancel{3^n}}$

$= \lim_{n\to\infty} \frac{3}{n+1} = 0 < 1$

$\boxed{\text{converges}}$

(b) $\sum_{n=1}^{\infty} \left(\frac{4n+2}{3n-1}\right)^n$.

Root Test: $\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \left(\left(\frac{4n+2}{3n-1}\right)^n\right)^{1/n} = \lim_{n\to\infty} \frac{4n+2}{3n-1}$

$= \lim_{n\to\infty} \left(\frac{4n+2}{3n-1}\right)\frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)}$

$= \lim_{n\to\infty} \frac{4+2/n}{3-1/n}$

$= \frac{4}{3} > 1$

$\boxed{\text{diverges}}$

(c) $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2+4}$.

Alternating Series Test:

(i) Show $\left\{\frac{1}{n^2+4}\right\}$ is decreasing: 3 methods!

$f(x) = \frac{1}{x^2+4} \Rightarrow f'(x) = \frac{0-(2x)}{(x^2+4)^2} = \frac{-2x}{(x^2+4)^2} < 0$ ✓

(ii) $\lim_{n\to\infty} \frac{1}{n^2+4} = 0$ ✓

So $\boxed{\text{converges}}$

2. Determine if the following series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}.$$

consider the series of absolute values: $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\sqrt{n}}\right| = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$

p-series with $p = 1/2 \leq 1$ diverges

Now consider $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ ← alternating series? So let's try to use the alternating series test:

(i) Show $\left\{\frac{1}{\sqrt{n}}\right\}$ is decreasing: 3 methods!

$f(x) = \frac{1}{\sqrt{x}} = x^{-1/2} \Rightarrow f'(x) = -\frac{1}{2}x^{-3/2} = \frac{-1}{2x^{3/2}} < 0$ ✓

(ii) $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$ ✓

So the series $\sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ converges, while the series $\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{\sqrt{n}}\right|$ diverges. Hence, the series $\boxed{\text{Conditionally converges}}$

3. Using the formula, set up a table and find the first THREE terms of the Maclaurin series for

$$f(x) = \frac{1}{1+3x} = (1+3x)^{-1}.$$

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$\frac{f^{(n)}(0)}{n!}$
0	$(1+3x)^{-1}$	1	$\frac{1}{0!} = 1$
1	$-(1+3x)^{-2}(3)$	-3	$\frac{-3}{1!} = -3$
2	$-(-2)(1+3x)^{-3}(3)^2$	18	$\frac{18}{2!} = 9$

Maclaurin series:

$f(0) + f'(0)x + \frac{f''(0)}{2!}x^2 + \cdots$

$$\boxed{1 + (-3)x + \left(\frac{18}{2!}\right)x^2 + \cdots}$$

OR

$$\boxed{1 - 3x + 9x^2 + \cdots}$$

4. Using the formula, set up a table and find the first THREE terms of the Taylor series about $x_0 = 2$ for

$$f(x) = e^{-x}.$$

n	$f^{(n)}(x)$	$f^{(n)}(2)$	$\frac{f^{(n)}(2)}{n!}$
0	e^{-x}	e^{-2}	$\frac{e^{-2}}{0!} = e^{-2}$
1	$-e^{-x}$	$-e^{-2}$	$\frac{-e^{-2}}{1!} = -e^{-2}$
2	e^{-x}	e^{-2}	$\frac{e^{-2}}{2!} = \frac{e^{-2}}{2}$

Taylor series (about $x_0=2$):

$f(2) + f'(2)(x-2) + \frac{f''(2)}{2!}(x-2)^2 + \cdots$

$$\boxed{e^{-2} + (-e^{-2})(x-2) + \left(\frac{e^{-2}}{2!}\right)(x-2)^2 + \cdots}$$

OR

$$\boxed{\frac{1}{e^2} - \frac{1}{e^2}(x-2) + \frac{1}{2e^2}(x-2)^2 + \cdots}$$

5. Find the radius and interval of convergence for the power series

$$\sum_{n=1}^{\infty} \frac{(x+3)^n}{n^2}.$$

Use the ratio test for absolute convergence first:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(x+3)^{n+1}}{(n+1)^2}\cdot\frac{n^2}{(x+3)^n}\right| = \lim_{n\to\infty}\left|\frac{\cancel{(x+3)^n}(x+3)n^2}{(n+1)^2\cancel{(x+3)^n}}\right|$$

$$= \lim_{n\to\infty}\frac{|x+3|\,n^2}{(n+1)^2}$$

$$= |x+3|\lim_{n\to\infty}\frac{(n^2)}{(n^2+2n+1)}\frac{(\frac{1}{n^2})}{(\frac{1}{n^2})}$$

$$= |x+3|\lim_{n\to\infty}\frac{1}{1+2/n+1/n^2}$$

$$= |x+3|\cdot 1 = |x+3|$$

Therefore, the series converges if $|x+3|<1$

$$-1 < x+3 < 1$$
$$-1-3 < x < 1-3$$
$$-4 < x < -2.$$

Now we need to check the end points!

at $x=-4$ (plug $x=-4$ into the power series) we get: $\displaystyle\sum_{n=1}^{\infty}\frac{(-4+3)^n}{n^2} = \sum_{n=1}^{\infty}\frac{(-1)^n}{n^2}$ alternating.

use the alternating series test (i) $\{\frac{1}{n^2}\}$ is decreasing: $f(x)=\frac{1}{x^2} \Rightarrow f'(x) = \frac{-2x}{(x^2)^2} < 0$ ✓

(ii) $\lim_{n\to\infty}\frac{1}{n^2} = 0$ ✓

converges

at $x=-2$ (plug $x=-2$ into the power series) we get: $\displaystyle\sum_{n=1}^{\infty}\frac{(-2+3)^n}{n^2} = \sum_{n=1}^{\infty}\frac{(1)^n}{n^2} = \sum_{n=1}^{\infty}\frac{1}{n^2}$

p-series $(p=2>1)$ converges

so the interval of convergence is $\boxed{-4 \le x \le -2}$ OR $\boxed{[-4,-2]}$

and the radius of convergence is $\boxed{R=1}$

6. Use known Maclaurin series to write the first THREE terms of the Maclaurin series for the following.

(a) $\dfrac{1}{1+3x^2} = \underbrace{\dfrac{1}{1-(-3x^2)}}$ and since $\dfrac{1}{1-x} = 1+x+x^2+\cdots$ we get

$$= \boxed{1+(-3x^2)+(-3x^2)^2+\cdots} \overset{\text{OR}}{=} \boxed{1-3x^2+9x^4+\cdots}$$

(b) $\displaystyle\int e^{x^3}\,dx.$

Since $e^x = 1+x+\dfrac{x^2}{2!}+\cdots$

$$e^{x^3} = 1+x^3+\frac{(x^3)^2}{2!}+\cdots = 1+x^3+\frac{x^6}{2!}+\cdots$$

Hence $\displaystyle\int e^{x^3}dx = \int\left(1+x^3+\frac{x^6}{2!}+\cdots\right)dx = \boxed{C+x+\frac{x^4}{4}+\frac{x^7}{7\cdot 2!}+\cdots}$