# TEST 4

Math 152 - Calculus II

**Score:** ________ out of 100

12/3/2013

**Name:** Key

**Read all of the following information before starting the exam:**

- You have 50 minutes to complete the exam.
- Show all work, clearly and in order, if you want to get full credit. Please make sure you read the directions for each problem. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Please $\boxed{\text{box/circle}}$ or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements.
- This test has 5 problems and is worth 100 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

1. Determine if the following series converge or diverge. Clearly state the test you are using to obtain your answer.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{n!}{7^n 9^{n+1}}$.

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{(n+1)!}{7^{n+1}9^{n+2}} \cdot \frac{7^n 9^{n+1}}{n!} = \lim_{n\to\infty} \frac{(n+1)}{7\cdot 9} = \infty$$

$\boxed{\text{diverges}}$ by Ratio Test.

(b) $\displaystyle\sum_{n=1}^{\infty} \left(\frac{1}{2} + \frac{1}{\ln(5n)}\right)^n$.

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \left[\left(\frac{1}{2} + \frac{1}{\ln(5n)}\right)^n\right]^{1/n}$$

$$= \lim_{n\to\infty} \left(\frac{1}{2} + \frac{1}{\ln(5n)}\right) = \frac{1}{2} + 0 = \frac{1}{2} < 1$$

$\boxed{\text{converges}}$ by Root Test.

(c) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{e^n + 1}$. alternating.

(a) Show $\left\{\frac{1}{e^n+1}\right\}$ is decreasing:

$$f(x) = \frac{1}{e^x+1} \longrightarrow f'(x) = \frac{(e^x+1)(0) - e^x}{(e^x+1)^2} = \frac{-e^x}{(e^x+1)^2} < 0.$$

(b) $$\lim_{n\to\infty} \frac{1}{e^n+1} = 0. \checkmark$$

series $\boxed{\text{converges}}$ by alt. series test.

2. Determine if the following series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=2}^{\infty}(-1)^n \frac{n}{n^2-1}.$$

Try Ratio Test for abs conv:

$$\lim_{n\to\infty}\left|\frac{(-1)^{n+1}(n+1)}{(n+1)^2-1}\cdot\frac{(n^2-1)}{(-1)^n n}\right| = \lim_{n\to\infty}\frac{n^3+n^2-n-1}{((n^2+2n+1)-1)n}$$

$$= \lim_{n\to\infty}\frac{n^3+n^2-n-1}{n^3+2n^2} = 1. \quad \underline{\text{NO INFO}}.$$

Look at $\sum |a_n| = \sum_{n=2}^{\infty}\left|\frac{(-1)^n\cdot n}{n^2-1}\right| = \sum_{n=2}^{\infty}\frac{n}{n^2-1}$

try comp. test (or limit comparison test)

$$\frac{n}{n^2-1} > \frac{n}{n^2} = \frac{1}{n}$$

Also, $\sum_{n=2}^{\infty}\frac{1}{n}$ diverges (harmonic series).

So original series diverges absolutely.

Look at $\sum a_n = \sum_{n=2}^{\infty}\frac{(-1)^n n}{n^2-1}$ ← alternating.

(a) Show: $\left\{\frac{n}{n^2-1}\right\}$ is decreasing:

$$f(x)=\frac{x}{x^2-1} \longrightarrow f'(x)=\frac{(x^2-1)\cdot 1 - x(2x)}{(x^2-1)^2} = \frac{x^2-1-2x^2}{(x^2-1)^2}$$

$$= \frac{-x^2-1}{(x^2-1)^2} \begin{matrix}\leftarrow \ominus \\ \leftarrow \oplus\end{matrix} < 0. \checkmark$$

(b) $\lim_{x\to\infty}\frac{x}{x^2-1} \overset{LH}{=} \lim_{x\to\infty}\frac{1}{2x} = 0.$ ✓

Series converges

So original series $\boxed{\text{conditionally converges}}$

3. Using the formula, **set up a table** and find the first THREE nonzero terms of the Maclaurin series for

$$f(x) = \sin\left(x + \frac{\pi}{2}\right).$$

Be sure to write out the series!

| $n$ | $f^{(n)}(x)$ | $f^{(n)}(0)$ | $\frac{f^{(n)}(0)}{n!}x^n$ |
|---|---|---|---|
| 0 | $\sin\left(x+\frac{\pi}{2}\right)$ | $\sin\left(\frac{\pi}{2}\right)=1$ | $\frac{1}{0!}x^0 = 1$ |
| 1 | $\cos\left(x+\frac{\pi}{2}\right)$ | 0 | 0 |
| 2 | $-\sin\left(x+\frac{\pi}{2}\right)$ | $-1$ | $\frac{-1}{2!}x^2$ |
| 3 | $-\cos\left(x+\frac{\pi}{2}\right)$ | 0 | 0 |
| 4 | $\sin\left(x+\frac{\pi}{2}\right)$ | 1 | $\frac{1}{4!}x^4$ |
| $\vdots$ | $\vdots$ | $\vdots$ | $\vdots$ |

$$\boxed{1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 + \cdots}$$

4. Using the formula, **set up a table** and find the first THREE nonzero terms of the Taylor series about $x_0 = 1$ for

$$f(x) = e^{-3x}.$$

Be sure to write out the series!

| $n$ | $f^{(n)}(x)$ | $f^{(n)}(1)$ | $\frac{f^{(n)}(1)}{n!}(x-1)^n$ |
|---|---|---|---|
| 0 | $e^{-3x}$ | $e^{-3}$ | $e^{-3}$ |
| 1 | $-3e^{-3x}$ | $-3e^{-3}$ | $-3e^{-3}(x-1)$ |
| 2 | $9e^{-3x}$ | $9e^{-3}$ | $\frac{9e^{-3}}{2!}(x-1)^2$ |

$$\boxed{e^{-3} + -3e^{-3}(x-1) + \frac{9e^{-3}}{2!}(x-1)^2 + \cdots}$$

5. Find the **radius of convergence** and **interval of convergence** for the power series

$$\sum_{n=1}^{\infty} \frac{(x-5)^n}{n^4}.$$

$$\lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty} \left| \frac{(x-5)^{n+1}}{(n+1)^4} \cdot \frac{n^4}{(x-5)^n} \right|$$

$$= \lim_{n\to\infty} |x-5| \left(\frac{n}{n+1}\right)^4 = |x-5| \cdot 1 = |x-5|$$

Series converges if $|x-5| < 1$

$$-1 < x-5 < 1$$

$$4 < x < 6$$

@ $x=4$ : $\sum_{n=1}^{\infty} \frac{(4-5)^n}{n^4} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^4}$ ← Alternating OR check for abs. convergence.

$\sum_{n=1}^{\infty} \left|\frac{(-1)^n}{n^4}\right| = \sum_{n=1}^{\infty} \frac{1}{n^4}$ p-series $p=4>1$ converges

@ $x=6$ : $\sum_{n=1}^{\infty} \frac{(6-5)^n}{n^4} = \sum_{n=1}^{\infty} \frac{1^n}{n^4} = \sum_{n=1}^{\infty} \frac{1}{n^4}$ p-series $p=4>1$ converges

**Interval of Convergence:** $\boxed{4 \le x \le 6}$ OR $[4,6]$

**Radius of Convergence:** $\boxed{R=1}$