TEST 3

Math 152 - Calculus II **Score:** ________ out of 100

4/5/2013 **Name:** key

Read all of the following information before starting the exam:

- You have 50 minutes to complete the exam.
- Show all work, clearly and in order, if you want to get full credit. Please make sure you read the directions for each problem. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Please $\boxed{\text{box/circle}}$ or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements.
- This test has 8 problems and is worth 100 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

1. Determine whether the sequence converges, and if so find its limit.

(a) $\left\{\dfrac{2n^4}{7n^4+2n^2-4}\right\}_{n=1}^{\infty}$

$$\lim_{n\to\infty} \frac{2n^4}{7n^4+2n^2-4} = \lim_{n\to\infty} \left(\frac{2n^4}{7n^4+2n^2-4}\right)\frac{\left(\frac{1}{n^4}\right)}{\left(\frac{1}{n^4}\right)} = \lim_{n\to\infty} \frac{2}{7+\frac{2}{n^2}-\frac{4}{n^4}} = \boxed{\frac{2}{7}}$$

sequence converges

(b) $\left\{\dfrac{\ln(n)}{e^n}\right\}_{n=1}^{\infty}$

$$\lim_{x\to\infty} \frac{\ln(x)}{e^x} \overset{L'H}{=} \lim_{x\to\infty} \frac{\left(\frac{1}{x}\right)}{e^x} = \lim_{x\to\infty} \frac{1}{xe^x} = 0$$

Hence, $\lim_{n\to\infty} \frac{\ln(n)}{e^n} = \boxed{0}$ sequence converges

(c) $\left\{\dfrac{(-1)^{n+1}}{n^3}\right\}_{n=1}^{\infty}$

$$\frac{-1}{n^3} \le \frac{(-1)^{n+1}}{n^3} \le \frac{1}{n^3}$$

$$\lim_{n\to\infty} \frac{-1}{n^3} = 0$$

AND $\lim_{n\to\infty} \frac{1}{n^3} = 0$. Therefore, by Squeeze Thm. $\lim_{n\to\infty} \frac{(-1)^{n+1}}{n^3} = \boxed{0}$

sequence converges

2. Show that the given sequence is strictly increasing or strictly decreasing.

$$\left\{\frac{n}{e^{2n}}\right\}_{n=1}^{\infty}$$

3 Methods!

$f(x) = \dfrac{x}{e^{2x}}$ so

$$f'(x) = \frac{e^{2x}(1) - x\,2e^{2x}}{(e^{2x})^2} = \frac{e^{2x}(1-2x)}{(e^{2x})^2} = \frac{1-2x}{e^{2x}} < 0$$

The sequence is strictly decreasing

3. Each series below is geometric. Determine both a and r. Then decide whether the series converges or diverges. If the series converges, then find its sum. If it diverges, write "NO SUM."

(a) $\displaystyle\sum_{k=1}^{\infty}\left(\frac{-1}{3\ln 5}\right)^{k-1}$

$a =$ 1

$r =$ $\dfrac{-1}{3\ln(5)}$

sum= $\dfrac{1}{1+\frac{1}{3\ln(5)}}$

(b) $\displaystyle\sum_{k=0}^{\infty}(-1)^{k+1}\frac{2^{4k}}{11^{k+1}}$ $= \displaystyle\sum_{K=0}^{\infty}(-1)^{k}(-1)^{1}\frac{16^{k}}{11^{k}\cdot 11^{1}} = \sum_{K=0}^{\infty}\left(\frac{-1}{11}\right)\left(\frac{-16}{11}\right)^{k}$

$a =$ $-1/11$

$r =$ $-16/11$

sum= NO SUM $\left(\text{since } |r| = \left|\frac{-16}{11}\right| = \frac{16}{11} > 1\right)$

4. Use the **Divergence Test** on each of the following to determine whether the given series diverges. If the test yields no conclusion, then be sure to say so. You must set up, evaluate, and interpret the correct limit to earn credit.

(a) $\displaystyle\sum_{n=1}^{\infty}\sin\left(\frac{1}{e^{n}}\right)$

$$\lim_{n\to\infty}\sin\left(\frac{1}{e^{n}}\right) = \sin(0) = 0 \quad \boxed{\text{NO INFO}}$$

(b) $\displaystyle\sum_{n=1}^{\infty}\frac{n}{4n-1}$

$$\lim_{n\to\infty}\frac{n}{4n-1} = \lim_{n\to\infty}\left(\frac{n}{4n-1}\right)\frac{\left(\frac{1}{n}\right)}{\left(\frac{1}{n}\right)} = \lim_{n\to\infty}\frac{1}{4-\frac{1}{n}} = \frac{1}{4} \neq 0$$

Hence, the series $\boxed{\text{diverges}}$

5. Use the **Integral Test** to determine whether the given series converges or diverges. Clearly identify the function $f(x)$ you are embedding the sequence of terms into. You may assume that $f(x)$ is positive, decreasing and continuous for $x \geq 1$, so you do not need to verify this. Just use the integral test and state your conclusion.

$$\sum_{n=1}^{\infty} \frac{\tan^{-1}(n)}{n^2+1}$$

$$f(x) = \frac{\tan^{-1}(x)}{x^2+1}$$

$$\int_1^{\infty} \frac{\tan^{-1}(x)}{x^2+1}\,dx = \lim_{t\to\infty} \int_1^t \frac{\tan^{-1}(x)}{x^2+1}\,dx$$

$$u = \tan^{-1}(x) \Rightarrow \begin{array}{l} u(1) = \tan^{-1}(1) = \pi/4 \\ u(t) = \tan^{-1}(t) \end{array}$$

$$\frac{du}{dx} = \frac{1}{x^2+1} \Rightarrow dx = (x^2+1)\,du$$

$$= \lim_{t\to\infty} \int_{\pi/4}^{\tan^{-1}(t)} u\,du$$

$$= \lim_{t\to\infty} \left[\frac{u^2}{2}\right]_{\pi/4}^{\tan^{-1}(t)} = \lim_{t\to\infty}\left[\frac{(\tan^{-1}(t))^2}{2} - \frac{(\pi/4)^2}{2}\right]$$

$$= \frac{(\pi/2)^2}{2} - \frac{(\pi/4)^2}{2} = \frac{3\pi^2}{32}$$

The integral converges $\Rightarrow$ the original series also $\boxed{\text{converges}}$

6. Use the **Comparison Test** to determine whether the given series converges or diverges. Clearly write down an inequality, determine the convergence or divergence of your comparison series $\sum_{n=1}^{\infty} b_n$, and then write your conclusion.

$$\sum_{n=1}^{\infty} \frac{\cos^4(n)}{5^n+1}$$

$$\frac{\cos^4(n)}{5^n+1} \leq \frac{1}{5^n}$$

Since $\sum_{n=1}^{\infty} \frac{1}{5^n} = \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)^n = \sum_{n=1}^{\infty} \left(\frac{1}{5}\right)\left(\frac{1}{5}\right)^{n-1}$ is geometric and converges (since $r = 1/5$ AND $|r| = 1/5 < 1$)

By Comparison test the original series also $\boxed{\text{converges}}$

7. Use the **Limit Comparison Test** to determine whether the given series converges or diverges. Clearly write down what a_n and b_n are, compute the appropriate limit, determine the convergence or divergence of your comparison series $\sum_{n=1}^{\infty} b_n$, and then write your conclusion.

$$\sum_{n=1}^{\infty} \frac{5n}{n^{3/2}+1}$$

$$a_n = \frac{5n}{n^{3/2}+1} \qquad b_n = \frac{5n}{n^{3/2}} = \frac{5}{n^{1/2}}$$

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{5n}{n^{3/2}+1}\right)}{\left(5/n^{1/2}\right)} = \lim_{n\to\infty} \frac{5n^{3/2}}{5n^{3/5}+5} = \lim_{n\to\infty} \left(\frac{n^{3/2}}{n^{3/2}+1}\right)\left(\frac{\left(1/n^{3/2}\right)}{\left(1/n^{3/2}\right)}\right)$$

$$= \lim_{n\to\infty} \frac{1}{1+1/n^{3/2}} = 1$$

finite and positive!

Since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} \frac{5}{n^{1/2}} = 5\sum_{n=1}^{\infty} \frac{1}{n^{1/2}}$ diverges (p-series $p = 1/2 \leq 1$)

by Limit Comparison Test the original series also $\boxed{\text{diverges}}$

8. Determine whether the following series converges or diverges. If the series converges, then **FIND ITS SUM**

$$\sum_{n=1}^{\infty} \left(\frac{1}{3^n} - \frac{1}{3^{n+1}}\right)$$

$\boxed{\text{SOL 1}}$

$$\sum_{n=1}^{\infty} \left(\frac{1}{3^n} - \frac{1}{3^{n+1}}\right)$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^{n-1} - \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^{n-1}$$

$$= \sum_{n=1}^{\infty} \left(\frac{1}{3}\right)\left(\frac{1}{3}\right)^{n-1} - \sum_{n=1}^{\infty} \left(\frac{1}{9}\right)\left(\frac{1}{3}\right)^{n-1}$$

converges: $a = 1/3$, $r = 1/3$ converges: $a = 1/9$, $r = 1/3$

$$= \frac{1/3}{1-1/3} - \frac{1/9}{1-1/3}$$

$$= \boxed{1/3}$$

series converges to this SUM

$\boxed{\text{SOL 2}}$

$$S_n = \left(\frac{1}{3^1} - \cancel{\frac{1}{3^2}}\right) + \left(\cancel{\frac{1}{3^2}} - \cancel{\frac{1}{3^3}}\right) + \left(\cancel{\frac{1}{3^3}} - \cancel{\frac{1}{3^4}}\right) + \cdots + \left(\cancel{\frac{1}{3^n}} - \frac{1}{3^{n+1}}\right)$$

$$= \frac{1}{3} - \frac{1}{3^{n+1}}$$

$$\lim_{n\to\infty} S_n = \lim_{n\to\infty} \left(\frac{1}{3} - \frac{1}{3^{n+1}}\right) = \frac{1}{3} - 0 = \boxed{\frac{1}{3}}$$

series converges to this sum