TEST 4

Math 152 - Calculus II | **Score:** ________ out of 100

4/26/2013 | **Name:** ______________________________

Read all of the following information before starting the exam:

- You have 50 minutes to complete the exam.
- Show all work, clearly and in order, if you want to get full credit. Please make sure you read the directions for each problem. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Please $\boxed{\text{box/circle}}$ or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements.
- This test has 5 problems and is worth 100 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

1. Determine if the following series converge or diverge. Clearly state the test you are using to obtain your answer.

(a) $\sum_{n=0}^{\infty} \frac{5^n}{(2n)!}$.

Try Ratio Test:

$$\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{5^{n+1}}{(2(n+1))!} \cdot \frac{(2n)!}{5^n}$$

$$= \lim_{n\to\infty} \frac{\cancel{5^n}\cdot 5^1 \cdot (2n)!}{(2n+2)!\ \cancel{5^n}}$$

$$= \lim_{n\to\infty} \frac{5\ \cancel{(2n)!}}{(2n+2)(2n+1)\cancel{(2n)!}}$$

$$= \lim_{n\to\infty} \frac{5}{(2n+2)(2n+1)} = 0 < 1$$

$\boxed{\text{series converges}}$

(b) $\sum_{n=1}^{\infty} \left(\frac{\tan^{-1}(2n)}{7\tan^{-1}(n)}\right)^n$.

Try Root Test:

$$\lim_{n\to\infty} (a_n)^{1/n} = \lim_{n\to\infty} \left(\left[\frac{\tan^{-1}(2n)}{7\tan^{-1}(n)}\right]^n\right)^{1/n}$$

$$= \lim_{n\to\infty} \frac{\tan^{-1}(2n)}{7\tan^{-1}(n)}$$

$$= \frac{\pi/2}{7\cdot \pi/2} = \frac{1}{7} < 1$$

$\boxed{\text{Series converges}}$

(c) $\sum_{n=3}^{\infty} (-1)^{n+1} \frac{\ln(n+2)}{n}$.

Try Alternating Series Test:

(a) Show $\left\{\frac{\ln(n+2)}{n}\right\}$ is decreasing

$$f(x) = \frac{\ln(x+2)}{x} \Rightarrow f'(x) = \frac{x\left(\frac{1}{x+2}\right) - \ln(x+2)}{x^2}$$

less than 1 for $x \geq 3$

greater than 1 for $x \geq 3$

$$= \frac{\frac{x}{x+2} - \ln(x+2)}{x^2} = \frac{1 - \frac{2}{x+2} - \ln(x+2)}{x^2} < 0$$

✓

(b) $\lim_{x\to\infty} \frac{\ln(x+2)}{x} \overset{L'H}{=} \lim_{x\to\infty} \frac{\left(\frac{1}{x+2}\right)}{1} = 0$ ✓

$\boxed{\text{series converges}}$

2. Determine if the following series is absolutely convergent, conditionally convergent, or divergent.

$$\sum_{n=2}^{\infty} \frac{(-1)^n}{\ln(n)}.$$

Try Ratio Test for Absolute Convergence:

$$\lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right| = \lim_{n\to\infty}\left|\frac{(-1)^{n+1}}{\ln(n+1)}\cdot\frac{\ln(n)}{(-1)^n}\right| = \lim_{n\to\infty}\frac{\ln(n)}{\ln(n+1)}$$

$$\lim_{x\to\infty}\frac{\ln(x)}{\ln(x+1)} \overset{L'H}{=} \lim_{x\to\infty}\frac{\left(\frac{1}{x}\right)}{\left(\frac{1}{x+1}\right)} = \lim_{x\to\infty}\frac{x+1}{x} = 1$$

NO INFO!

Look at $\sum_{n=2}^{\infty}\left|\frac{(-1)^n}{\ln(n)}\right| = \sum_{n=2}^{\infty}\frac{1}{\ln(n)}$. Notice that $n \geq \ln(n)$ for $n \geq 2$ (really for $n>0$)

Hence,

Try Comparison test

$$\frac{1}{\ln(n)} \geq \frac{1}{n}$$

Since $\sum_{n=2}^{\infty}\frac{1}{n}$ diverges (harmonic series $\sum_{n=1}^{\infty}\frac{1}{n}$ diverges, so does $\sum_{n=2}^{\infty}\frac{1}{n}$)

Hence, $\sum_{n=2}^{\infty}\frac{1}{\ln(n)}$ diverges $\Rightarrow$ $\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln(n)}$ diverges absolutely.

Now look at the original series $\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln(n)}$

Try Alternating Series test:

(a) Show $\left\{\frac{1}{\ln(n)}\right\}$ is decreasing: $f(x)=\frac{1}{\ln(x)} \Rightarrow f'(x)=\frac{-\left(\frac{1}{x}\right)}{(\ln(x))^2} < 0$ ✓

(b) $\lim_{n\to\infty}\frac{1}{\ln(n)} = 0$ ✓

Hence, the original series $\sum_{n=2}^{\infty}\frac{(-1)^n}{\ln(n)}$ is $\boxed{\text{conditionally convergent}}$

3. Using the formula, set up a table and find the first THREE nonzero terms of the Maclaurin series for

$$f(x) = \ln(1+x).$$

Be sure to write out the series!

n	$f^{(n)}(x)$	$f^{(n)}(0)$	$f^{(n)}(0)/n!$
0	$\ln(1+x)$	$\ln(1) = 0$	0
1	$\frac{1}{1+x} = (1+x)^{-1}$	1	$1/1! = 1$
2	$-(1+x)^{-2}$	-1	$-1/2! = -\frac{1}{2}$
3	$(-1)(-2)(1+x)^{-3}$	2	$2/3! = \frac{1}{3}$

Maclaurin Series:

$$1\cdot x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + \cdots$$

$$= \boxed{x - \frac{x^2}{2} + \frac{x^3}{3} + \cdots}$$

4. Using the formula, **set up a table** and find the first THREE nonzero terms of the Taylor series about $x_0 = 4$ for

$$f(x) = \sqrt{x}.$$

Be sure to write out the series!

n	$f^{(n)}(x)$	$f^{(n)}(4)$	$f^{(n)}(4)/n!$
0	$\sqrt{x} = x^{1/2}$	$\sqrt{4} = 2$	$\frac{2}{0!} = 2$
1	$\frac{1}{2} x^{-1/2}$	$\frac{1}{2}(4)^{-1/2} = \frac{1}{2 \cdot 2} = \frac{1}{4}$	$\frac{1}{4 \cdot 1!} = \frac{1}{4}$
2	$(\frac{1}{2})(-\frac{1}{2}) x^{-3/2}$	$(\frac{1}{2})(-\frac{1}{2})(\frac{1}{8}) = -\frac{1}{32}$	$\frac{-1}{32 \cdot 2!} = \frac{-1}{64}$

Taylor Series about $x_0 = 4$: $\boxed{2 + \frac{1}{4}(x-4) - \frac{1}{64}(x-4)^2 + \cdots}$

5. Find the **radius of convergence** and **interval of convergence** for the power series

$$\sum_{n=1}^{\infty} \frac{(x-2)^n}{n}.$$

use Ratio Test for Absolute Convergence:

$$\lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right| = \lim_{n\to\infty} \left| \frac{(x-2)^{n+1}}{n+1} \cdot \frac{n}{(x-2)^n} \right| = \lim_{n\to\infty} \left| \frac{\cancel{(x-2)^n}(x-2) \cdot n}{(n+1)\cancel{(x-2)^n}} \right|$$

$$= \lim_{n\to\infty} \frac{|x-2| \cdot n}{n+1}$$

$$= |x-2| \lim_{n\to\infty} \frac{n}{n+1}$$

$$= |x-2|$$

So the series converges if $|x-2| < 1$
diverges if $|x-2| > 1$
NO INFO if $|x-2| = 1$

That is, converges if $-1 < x-2 < 1 \implies 1 < x < 3$

test $x=1$ and $x=3$!

At $x=1$: $\sum_{n=1}^{\infty} \frac{(1-2)^n}{n} = \sum_{n=1}^{\infty} \frac{(-1)^n}{n}$ ← use alternating series test.

(a) Show $\{\frac{1}{n}\}$ is decreasing: $f(x) = \frac{1}{x} \Rightarrow f'(x) = \frac{-1}{x^2} < 0$ ✓

(b) $\lim_{n\to\infty} \frac{1}{n} = 0$ ✓

So the power series converges at $x=1$.

At $x=3$: $\sum_{n=1}^{\infty} \frac{(3-2)^n}{n} = \sum_{n=1}^{\infty} \frac{1^n}{n} = \sum_{n=1}^{\infty} \frac{1}{n}$ ← Diverges! (Harmonic Series)

So the power series diverges at $x=3$

Therefore, the interval of convergence is $\boxed{[1, 3)}$ OR $\boxed{1 \le x < 3}$

and the radius of convergence is $\boxed{R = 1}$