POLYNOMIAL LONG DIVISION & INTEGRATION

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Example 0.1. Evaluate:

$$\int \frac{x^3 - 5}{x + 1} dx.$$

Notice that the degree of the numerator is 3 and the degree of the denominator is 1 so we must perform polynomial long division.

$$\begin{array}{r|rrrr}
 & x^2 & -\,x & +\,1 & \\
\hline
x+1 & x^3 & & & -\,5 \\
 & -\,x^3 - x^2 & & & \\
\hline
 & & -\,x^2 & & \\
 & & x^2 + x & & \\
\hline
 & & & x & -\,5 \\
 & & & -\,x & -\,1 \\
\hline
 & & & & -\,6
\end{array}$$

Therefore,

$$\begin{aligned}
\int \frac{x^3 - 5}{x + 1} dx &= \int \left(x^2 - x + 1 - \frac{6}{x + 1} \right) dx \\
&= \frac{1}{3}x^3 - \frac{1}{2}x^2 + x - 6 \ln |x + 1| + C.
\end{aligned}$$

Example 0.2. Evaluate:

$$\int \frac{x^4 - 1}{x^2 + 1} dx.$$

Notice that the degree of the numerator is 4 and the degree of the denominator is 2 so we must perform polynomial long division, or do we!?

SOLUTION 1: Notice that we can factor and reduce the problem:

$$\int \frac{x^4 - 1}{x^2 + 1} dx = \int \frac{\cancel{(x^2 + 1)}(x^2 - 1)}{\cancel{x^2 + 1}} dx = \int (x^2 - 1) dx = \frac{1}{3}x^3 - x + C.$$

SOLUTION 2: We can still use polynomial long division to solve the problem:

$$\begin{array}{r|l} & x^2 - 1 \\ \hline x^2+1 & x^4 \qquad - 1 \\ & -x^4 - x^2 \\ \hline & \quad -x^2 - 1 \\ & \quad\ \ x^2 + 1 \\ \hline & \qquad\quad 0 \end{array}$$

Therefore,

$$\begin{aligned} \int \frac{x^4-1}{x^2+1} dx &= \int \left(x^2 - 1 + \frac{0}{x^2+1} \right) dx \\ &= \int (x^2-1) dx \\ &= \frac{1}{3}x^3 - x + C. \end{aligned}$$

Example 0.3. Perform polynomial long division on:

$$\frac{x^2-5}{x-4}.$$

SOLUTION:

$$\begin{array}{r|l} & x \quad + 4 \\ \hline x-4 & x^2 \qquad - 5 \\ & -x^2 + 4x \\ \hline & \quad 4x \ - 5 \\ & \quad -4x + 16 \\ \hline & \qquad\quad 11 \end{array}$$

Therefore,

$$\frac{x^2-5}{x-4} = x + 4 + \frac{11}{x-4}.$$

Example 0.4. Perform polynomial long division on:

$$\frac{x^2+4}{x+2}.$$

SOLUTION:

$$\begin{array}{r|l} & x - 2 \\ \hline x+2 & x^2 \qquad + 4 \\ & -x^2 - 2x \\ \hline & \quad -2x + 4 \\ & \quad\ \ 2x + 4 \\ \hline & \qquad\quad 8 \end{array}$$

Therefore,

$$\frac{x^2+4}{x+2} = x - 2 + \frac{8}{x+2}.$$

Example 0.5. Perform polynomial long division on:

$$\frac{x^6 - 2x^3 + 1}{x + 2}.$$

SOLUTION:

$$
\begin{array}{r|rrrrrrr}
 & x^5 & -2x^4 & +4x^3 & -10x^2 & +20x & -40 & \\
\hline
x+2 & x^6 & & & -2x^3 & & & +1 \\
 & -x^6 & -2x^5 & & & & & \\
\cline{2-3}
 & & -2x^5 & & & & & \\
 & & 2x^5 & +4x^4 & & & & \\
\cline{3-4}
 & & & 4x^4 & -2x^3 & & & \\
 & & & -4x^4 & -8x^3 & & & \\
\cline{4-5}
 & & & & -10x^3 & & & \\
 & & & & 10x^3 & +20x^2 & & \\
\cline{5-6}
 & & & & & 20x^2 & & \\
 & & & & & -20x^2 & -40x & \\
\cline{6-7}
 & & & & & & -40x & +1 \\
 & & & & & & 40x & +80 \\
\cline{7-8}
 & & & & & & & 81
\end{array}
$$

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