Math 152 - Fall 2012

1. Use known Maclaurin series to write the first three terms of the Maclaurin series for the following:
   (a) xe^{-5x}
   (b) $\dfrac{1}{1+7x}$
   (c) $\dfrac{x}{1+7x}$
   (d) $\dfrac{d}{dx}\left(\dfrac{\cos(x)-1}{x}\right)$
   (e) $\int e^{x^2}\,dx$

2. Use known Maclaurin series to find the following limit:

$$\lim_{x\to 0}\frac{\sin(x)}{x}$$

(1)

(a) $x e^{-5x}$

since $e^{x} = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$

$e^{-5x} = \sum_{n=0}^{\infty} \frac{(-5x)^n}{n!} = 1 + (-5x) + \frac{(-5x)^2}{2!} + \frac{(-5x)^3}{3!} + \cdots$

$= 1 - 5x + \frac{25x^2}{2!} + \frac{-125x^3}{3!} + \cdots$

$x e^{-5x} = x\left(\sum_{n=0}^{\infty} \frac{(-5x)^n}{n!}\right) = x\left(1 - 5x + \frac{25x^2}{2!} - \frac{125x^3}{3!} + \cdots\right)$

$= \boxed{x - 5x^2 + \frac{25x^3}{2!} + \cdots}$

(b) $\frac{1}{1+7x}$

since $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \cdots$

$\frac{1}{1+7x} = \frac{1}{1-(-7x)} = \sum_{n=0}^{\infty} (-7x)^n = 1 - 7x + (-7x)^2 + \cdots$

$= \boxed{1 - 7x + 49x^2 + \cdots}$

(c) $\frac{x}{1+7x}$

since $\frac{1}{1+7x} = 1 - 7x + 49x^2 + \cdots$

$\frac{x}{1+7x} = x\left(\frac{1}{1+7x}\right) = x\left(1 - 7x + 49x^2 + \cdots\right)$

$= \boxed{x - 7x^2 + 49x^3 + \cdots}$

(d) $\frac{d}{dx}\left(\frac{\cos(x) - 1}{x}\right)$

since $\cos(x) = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$

$\frac{\cos(x)-1}{x} = \frac{1}{x}\left(\cos(x) - 1\right) = \frac{1}{x}\left(-\frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots\right)$

$= -\frac{x}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} + \cdots$

so $$\frac{d}{dx}\left(\frac{\cos(x)-1}{x}\right) = \frac{d}{dx}\left(\frac{-x}{2!} + \frac{x^3}{4!} - \frac{x^5}{6!} + \cdots\right)$$

$$= \boxed{\frac{-1}{2!} + \frac{3x^2}{4!} - \frac{5x^4}{6!} + \cdots}$$

(e) $$\int e^{x^2}\,dx$$

Since $$e^x = 1 + x + \frac{x^2}{2!} + \cdots$$

$$e^{x^2} = 1 + x^2 + \frac{(x^2)^2}{2!} + \cdots$$

$$= 1 + x^2 + \frac{x^4}{2!} + \cdots$$

so

$$\int e^{x^2}dx = \boxed{C + x + \frac{x^3}{3} + \frac{x^5}{5\cdot 2!} + \cdots}$$

(2) $\lim_{x\to 0} \frac{\sin(x)}{x}$:

Since $$\sin(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots$$

$$\frac{\sin(x)}{x} = \frac{1}{x}\left(x - \frac{x^3}{3!} + \frac{x^5}{5!} + \cdots\right)$$

$$= 1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots$$

$$\lim_{x\to 0} \frac{\sin(x)}{x} = \lim_{x\to 0}\left(1 - \frac{x^2}{3!} + \frac{x^4}{5!} + \cdots\right) = 1 - 0 + 0 + \cdots$$

$$= \boxed{1}$$