TEST 3

Math 271 - Differential Equations **Score:** ________ out of 100

Name: Key

Read all of the following information before starting the exam:

- You have 50 minutes to complete the exam.
- Show all work, clearly and in order, if you want to get full credit. Please make sure you read the directions for each problem. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Please box/circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements.
- This test has 5 problems and is worth 100 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

Topics

1. Form for y_p using the Method of Undetermined coef.
2. Cauch-Euler, homogeneous Equations (auxiliary/characteristic equations)
3. Method of Undetermined Coef. Nonhomogeneous Linear Equation with constant coef.
4. Variation of Parameters Nonhomogeneous Cauchy-Euler Equation
5. Spring Problem (Nonhomogeneous Linear Equation with constant coef.

1. Suppose you have a 2nd order linear differential equations with constant coefficients:

$$ay'' + by' + cy = g(x).$$

Using the method of undetermined coefficients write the FORM for the particular solution (y_p) using the given value for $g(x)$ and the general solution of the associated homogeneous equation (y_c). Do NOT try to solve for the unknown constants, just write the form.

(a) $g(x) = x^2 - 12e^{2x}$ and $y_c = C_1e^{-x} + C_2xe^{-x}$. so

Form of y_p: $\boxed{Ax^2 + Bx + C + De^{2x}}$

(b) $g(x) = \sin(3x)$ and $y_c = C_1\sin(x) + C_2\cos(x)$. so

Form of y_p: $\boxed{A\sin(3x) + B\cos(3x)}$

(c) $g(x) = 4e^x\sin 2x$ and $y_c = C_1e^x\sin(2x) + C_2e^x\cos(2x)$. so

Form of y_p: $\boxed{Axe^x\sin(2x) + Bxe^x\cos(2x)}$

2. Solve the following homogeneous Cauchy-Euler Equations:

(a) $x^2y'' - 3xy' = 0$

$m^2 + (-3-1)m + 0 = 0$

$m^2 - 4m = 0$

$m(m-4) = 0$

$m = 0 \mid m = 4$

General Solution: $\boxed{y = C_1x^0 + C_2x^4 = C_1 + C_2x^4}$

(b) $4x^2y'' + y = 0$

$4m^2 + (0-4)m + 1 = 0$

$4m^2 - 4m + 1 = 0$

$m = \dfrac{4 \pm \sqrt{16-4(4)(1)}}{2(4)} = \dfrac{4 \pm \sqrt{0}}{2(4)} = \dfrac{1}{2}$ $\quad m = 1/2 \mid m = 1/2$

General Solution: $\boxed{C_1x^{1/2} + C_2\ln(x)\,x^{1/2}}$

(c) $3x^2y'' + 6xy' + y = 0$

$3m^2 + (6-3)m + 1 = 0$

$3m^2 + 3m + 1 = 0$

$m = \dfrac{-3 \pm \sqrt{9-4(3)(1)}}{2(3)} = \dfrac{-3 \pm \sqrt{-3}}{6} = -\dfrac{1}{2} \pm \dfrac{\sqrt{3}}{6}i$ $\quad \alpha = -1/2,\ \beta = \dfrac{\sqrt{3}}{6}$

General Solution: $\boxed{C_1x^{-1/2}\cos\left(\frac{\sqrt{3}}{6}\ln(x)\right) + C_2x^{-1/2}\sin\left(\frac{\sqrt{3}}{6}\ln(x)\right)}$

3. Solve the following differential equation using the method of undetermined coefficients:

$$y'' - 16y = 2e^{4x}$$

$\boxed{\text{Find } y_c}$:

$$y'' - 16y = 0$$
$$m^2 - 16 = 0$$
$$(m-4)(m+4) = 0$$
$$m = 4 \quad | \quad m = -4$$
$$y_c = C_1 e^{4x} + C_2 e^{-4x}$$

$\boxed{\text{Find } y_p}$:

Form looking at $g(x) = 2e^{4x}$:

$$y_p = Ae^{4x}$$

Adjust looking at y_c? Yes! e^{4x} is part of y_c so

$$y_p = Axe^{4x}$$

$$y_p' = Ae^{4x} + 4Axe^{4x}$$
$$y_p'' = 4Ae^{4x} + (4Ax)(4e^{4x}) + (4A)e^{4x}$$
$$= 8Ae^{4x} + 16Axe^{4x}$$

Hence, $y_p'' - 16y_p = 2e^{4x}$ becomes:

$$8Ae^{4x} + 16Axe^{4x} - 16Axe^{4x} = 2e^{4x}$$
$$8Ae^{4x} = 2e^{4x}$$
$$8A = 2$$
$$A = \frac{2}{8} = \frac{1}{4}$$

So

$$y_p = \frac{1}{4}xe^{4x}$$

$\boxed{\text{General Solution}}$: $y = y_c + y_p$. So

General Solution: $\boxed{y = C_1 e^{4x} + C_2 e^{-4x} + \frac{1}{4}xe^{4x}}$

4. Solve the following Cauchy-Euler Equation (please simplify you final answer):

$$x^2y'' - xy' + y = 2x$$

$\boxed{\text{Find } y_c:}$

$$x^2y'' - xy' + y = 0$$
$$m^2 + (-1-1)m + 1 = 0$$
$$m^2 - 2m + 1 = 0$$
$$(m-1)(m-1) = 0$$
$$m = 1 \mid m = 1$$
$$y_c = C_1 \underbrace{x}_{y_1} + C_2 \underbrace{x\ln(x)}_{y_2}$$

$\boxed{\text{Find } y_p}$: Standard Form: $y'' - \frac{1}{x}y' + \frac{1}{x^2}y = \frac{2x}{x^2} = \underbrace{\frac{2}{x}}_{f(x)}$

Wronskian: $W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} x & x\ln(x) \\ 1 & x(\frac{1}{x}) + \ln(x) \end{vmatrix} = \begin{vmatrix} x & x\ln(x) \\ 1 & 1+\ln(x) \end{vmatrix}$

$= x(1+\ln(x)) - x\ln(x)$

$= x$

$$u_1 = \int \frac{-y_2 f(x)}{W}dx = \int \frac{-x\ln(x)\left(\frac{2}{x}\right)}{x}dx = -2\int \frac{\ln(x)}{x}dx$$

$u = \ln(x) \Rightarrow \frac{du}{dx} = \frac{1}{x}$

$\Rightarrow dx = x\,du$

$$= -2\int \frac{u}{x} \cdot x\,du$$
$$= -2\int u\,du = -\frac{2u^2}{2} = -u^2 = -(\ln(x))^2$$

$$u_2 = \int \frac{y_1 f(x)}{W}dx = \int \frac{x\left(\frac{2}{x}\right)}{x}dx = \int \frac{2}{x}dx = 2\ln(x)$$

$$y_p = u_1y_1 + u_2y_2 = -(\ln(x))^2 \cdot x + 2\ln(x)\cdot x\ln(x)$$
$$y_p = x(\ln(x))^2$$

$\boxed{\text{General Solution}}$: $y = y_p + y_p$ so →

Simplified General Solution: $\boxed{C_1x + C_2x\ln(x) + x(\ln(x))^2}$

5. Suppose we have a spring system with a mass of $m = \frac{1}{2}$ attached to a spring with $k = 6$ (spring constant). The mass is released from rest ($x'(0) = 0$) at a position of 2 units above the equilibrium position ($x(0) = -2$). The motion is damped ($\beta = \frac{1}{2}$) and is being driven by an external force ($f(t) = 10\cos(3t)$). Solve each of the following parts so we can find the motion of the spring.

(a) Write out the differential equation $m\frac{d^2x}{dt^2} + \beta\frac{dx}{dt} + kx = f(t)$.

$$\boxed{\frac{1}{2}\frac{d^2x}{dt^2} + \frac{1}{2}\frac{dx}{dt} + 6x = 10\cos(3t)}$$

(b) Find x_c (the solution to the associated homogeneous equation).

$$\frac{1}{2}x'' + \frac{1}{2}x + 6x = 0$$

$$\frac{1}{2}m^2 + \frac{1}{2}m + 6 = 0$$

$$m^2 + m + 12 = 0$$

$$m = \frac{-1 \pm \sqrt{1-4(1)(12)}}{2} = \frac{-1 \pm \sqrt{-47}}{2}$$

$$= -\frac{1}{2} \pm \frac{\sqrt{47}}{2} i$$

$$x_c = \boxed{C_1 e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{47}}{2}t\right) + C_2 e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{47}}{2}t\right)}$$

(c) The particular solution x_p is

$$x_p = \frac{10}{3}\Big(\cos(3t) + \sin(3t)\Big).$$

Now with the initial conditions we find

$$c_1 = \frac{-4}{3} \qquad c_2 = \frac{-64}{3\sqrt{47}}.$$

Write the exact solution:

$$x = \boxed{\frac{-4}{3} e^{-\frac{1}{2}t} \cos\left(\frac{\sqrt{47}}{2}t\right) + \left(\frac{-64}{3\sqrt{47}}\right) e^{-\frac{1}{2}t} \sin\left(\frac{\sqrt{47}}{2}t\right) + \frac{10}{3}\left(\cos(3t) + \sin(3t)\right)}$$