Math 271 - Differential Equations - Test 4

Nathan Reff
reff@alfred.edu
Alfred University
Department of Mathematics
http://people.alfred.edu/~reff/MATH271/

Analytic at $x = a$

A function $f(x)$ is **analytic at** $x = a$ if there exists a power series representation for $f(x)$ centered at a (with positive radius of convergence). That is, we can write:

$$f(x) = \sum_{n=0}^{\infty} c_n (x-a)^n,$$

where the radius of convergence for the power series is $R > 0$.
Note 1: polynomials are analytic everywhere!

Note 2: rational functions are analytic everywhere except at the roots of the denominator.

Ordinary and Singular Points

Suppose you want to solve a 2nd order homogeneous linear differential equation

$$a_2(x)y'' + a_1(x)y' + a_0(x)y = 0. \tag{0.1}$$

Put into standard form:

$$y'' + P(x)y' + Q(x)y = 0. \tag{0.2}$$

A point $x = x_0$ is an **ordinary point** of the differential equation (**??**) if both $P(x)$ and $Q(x)$ (from the standard form (**??**)) are analytic at x_0. Otherwise, the point $x = x_0$ is a **singular point**.

Regular and Irregular Singular Points

A singular point $x = x_0$ is a **regular singular point** of the differential equation (**??**) if

$$p(x) = (x - x_0)P(x), \text{ and}$$
$$q(x) = (x - x_0)^2 Q(x),$$

are both analytic at x_0. Otherwise, the singular point $x = x_0$ is an **irregular singular point**.

Series Solutions - Ordinary Points

If $x = x_0$ is an ordinary point of the differential equation (**??**), then we can always find two linearly independent solutions with power series representations centered at x_0 (with a nonzero radius of convergence). That is,

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^n.$$

While this is the general theory, we will usually solve problems where $x_0 = 0$ is an ordinary point.

Series Solutions - Regular Singular Points

If $x = x_0$ is a regular singular point of the differential equation (**??**), then we can always at least one nonzero solution of the form:

$$y = \sum_{n=0}^{\infty} c_n (x - x_0)^{n+r}.$$

While this is the general theory, we will usually solve problems where $x_0 = 0$ is a regular singular point.

Finding a power series solution about the ordinary point $x_0 = 0$

1. Check that $x_0 = 0$ is an ordinary point (unless you are told so).
2. Assume $y = \sum_{n=0}^{\infty} c_n x^n$.
3. Find y' and y''
4. Plug y, y' and y'' into the differential equation (**??**).
5. Add the resulting series to get a single power series equal to 0. There are three main steps to this process:
   (a) Bring in any coefficients to get a sum of individual series.
   (b) Check the *phase*! You want all the series to start with the same power of x. Pull terms out of any series that starts at lower powers until there are series *in phase*.
   (c) *Shift* the remaining series to all start at the same index value.
   (d) Now you may write the result as a single power series (and possibly some extra terms out of the sum) equal to 0.
6. The identity property can be used to set the coefficients of the powers of x to 0. This gives you a system of infinitely many linear equations. Solve for the highest order c_k that occurs to get your recurrence relation.
7. Set up a table to solve the above equations for c_0, c_1, c_2, ... etc. in terms of a small number of the c_is. (Usually everything will be terms of c_0 and c_1, or perhaps c_1 and c_2.)
8. Group your final answer into $y = C_A(\text{series 1}) + C_B(\text{series 2})$. Your solutions are:
   $$y_1 = \text{series 1},$$
   $$y_2 = \text{series 2}.$$
9. (Rare last step) Write a closed form for the series solutions.

Finding a power series solution about the regular singular point $x_0 = 0$

1. Check that $x_0 = 0$ is a regular singular point (unless you are told so).
2. Assume $y = \sum_{n=0}^{\infty} c_n x^{n+r}$.
3. Find y' and y''
4. Plug y, y' and y'' into the differential equation (**??**).
5. Add the resulting series to get a single series equal to 0. There are three main steps to this process:
   (a) Bring in any coefficients to get a sum of individual series.
   (b) Check the *phase*! You want all the series to start with the same power of x. Pull terms out of any series that starts at lower powers until there are series *in phase*.
   (c) *Shift* the remaining series to all start at the same index value.
   (d) Now you may write the result as a single series (and possibly some extra terms out of the sum) equal to 0.
6. The identity property can be used to set the coefficients of the powers of x to 0. This gives you a system of infinitely many linear equations. Solve for the highest order c_k that occurs to get your recurrence relation.
7. Find the **indicial equation** and solve for r. This the the equation that came from the coefficient of the lowest power of x (usually a quadratic in r).
8. For each value of r set up a table to solve the above equations for c_0, c_1, c_2, ... etc. in terms of a small number of the c_is. (Usually everything will be terms of c_0 and c_1, or perhaps c_1 and c_2.)
9. Group your final answer into $y = C_A(\text{series 1}) + C_B(\text{series 2})$. Don't forget to match the r with the series! Your solutions are:
   $$y_1 = \text{series 1},$$
   $$y_2 = \text{series 2}.$$
10. (Rare last step) Write a closed form for the series solutions.

Note: in general 2 linearly independent solutions are not always found, see the notes for the general theory.