TEST 1

Math 271 - Differential Equations

Score: ________ out of 100

2/12/2013

Name: Key

Read all of the following information before starting the exam:

- You have 50 minutes to complete the exam.
- Show all work, clearly and in order, if you want to get full credit. Please make sure you read the directions for each problem. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Please box/circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements.
- This test has 6 problems and is worth 100 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

1. Fill in the following table with the missing classification information:

+6

ODE	order	linear/nonlinear
$xy''' - \sin(x)y' = x\ln(x)$	3	linear
$(xy+2)y'' = y^5$	2	nonlinear
$\frac{dR}{dt} = 2014R$	1	linear

2. (a) Verify that $y = Ce^{x-x^2}$ is a one-parameter general solution to the differential equation

$$y' + (2x-1)y = 0$$

$y = Ce^{x-x^2}$

$y' = Ce^{x-x^2}(1-2x)$

+10

so $y' + (2x-1)y$

$= Ce^{x-x^2}(1-2x) + (2x-1)Ce^{x-x^2}$

$= -Ce^{x-x^2}(2x-1) + (2x-1)Ce^{x-x^2}$

$= 0.$ ✓

(b) Use part (a) to find a solution to the initial value problem (IVP) consisting of the differential equation $y' + (2x-1)y = 0$ and the initial condidtion $y(1) = 6$.

+10

$y(1) = 6 = Ce^{1-1^2} = Ce^0 = C$

so $C = 6.$

$\boxed{y = 6e^{x-x^2}}$

3. Find a solution to the following initial-value problem:

$$\frac{dy}{dx} = \frac{y^2+5y+6}{\sqrt{1-x^2}}, \qquad y(0) = 2.$$

$\underline{+20}$

+4

$$\int \frac{dy}{y^2+5y+6} = \int \frac{dx}{\sqrt{1-x^2}}$$

+3

$$\int \frac{1}{(y+3)(y+2)}\, dy = \sin^{-1}(x) + C$$

partial fractions:

$$\frac{1}{(y+3)(y+2)} = \frac{A}{y+3} + \frac{B}{y+2}$$

$$= \frac{A(y+2) + B(y+3)}{(y+3)(y+2)}$$

$$1 = A(y+2) + B(y+3)$$

$$1 = Ay + 2A + By + 3B$$

so $1 = 2A + 3B$ $\quad$ $0 = A + B$

+5 $\quad 1 = -2B + 3B$ $\quad$ $A = -B$

$\underline{B = 1} \longrightarrow \underline{A = -1}$

Now the left side becomes

$$\int \left(\frac{-1}{y+3} + \frac{1}{y+2} \right) dy = \sin^{-1}(x) + C$$

+3 $\quad -\ln|y+3| + \ln|y+2| = \sin^{-1}(x) + C$

+5 substituting initial conditions $y(0) = 2$:

$$-\ln|5| + \ln|4| = \sin^{-1}(0) + C$$

$$\ln(4/5) = 0 + C$$

$$C = \ln(4/5)$$

Implicit (or Explicit) Solution: $\boxed{\ln|y+2| - \ln|y+3| = \sin^{-1}(x) + \ln(4/5)}$

(find explicit, which is harder)

OR $\longrightarrow$ $e^{\ln\left|\frac{y+2}{y+3}\right|} = e^{\sin^{-1}(x) + C}$

$$\left|\frac{y+2}{y+3}\right| = e^{\sin^{-1}(x)} \underbrace{e^{C}}_{D}$$

$$\frac{y+2}{y+3} = E e^{\sin^{-1}(x)}$$

$$1 - \frac{1}{y+3} = E e^{\sin^{-1}(x)}$$

$$1 - E e^{\sin^{-1}(x)} = \frac{1}{y+3}$$

$$\longleftarrow \quad y = \frac{1}{1 - E e^{\sin^{-1}(x)}} - 3$$

so $2 = \frac{1}{1-E} - 3$

$\longleftarrow \quad 5 = \frac{1}{1-E}$

$\longleftarrow \quad 1 - E = \frac{1}{5}$, $\quad E = 4/5$

explicit:

$$\boxed{y = \frac{1}{1 - \frac{4}{5} e^{\sin^{-1}(x)}} - 3}$$

+20

4. (a) Find an explicit solution of:

$$x\frac{dy}{dx} + y = 2x\ln(x).$$

Be sure to clearly label steps to maximize your score.

1st order linear:

+3 Standard Form: $\frac{dy}{dx} + \frac{1}{x}y = 2\ln(x)$

+5 Integrating Factor: $e^{\int P(x)dx} = e^{\int 1/x\,dx} = e^{\ln|x|} = |x| = x$ (if $x>0$)

+2 Multiply: $x\left(\frac{dy}{dx} + \frac{1}{x}y\right) = 2x\ln(x)$

$$\frac{d}{dx}[xy] = 2x\ln(x)$$

Integrate:

$$xy = 2\int x\ln(x)\,dx$$

LIATE ($\ln(x)$, x)

$u = \ln(x)$ $\quad dv = x$

$du = \frac{1}{x}$ $\quad v = \frac{x^2}{2}$

$$xy = 2\left(\frac{\ln(x)x^2}{2} - \int\left(\frac{x^2}{2}\right)\left(\frac{1}{x}\right)dx\right)$$

$$xy = \ln(x)x^2 - \int x\,dx$$

+8 $$xy = x^2\ln(x) - \frac{x^2}{2} + C$$

+2 $$y = x\ln(x) - \frac{x}{2} + \frac{C}{x}$$

Explicit Solution: $\boxed{y = x\ln(x) - \frac{x}{2} + \frac{C}{x}}$

+5 (b) Give the largest interval over which the general solution is defined. $\boxed{x>0 \text{ OR } (0,\infty)}$

+5 (c) Are there any transient terms in the general solution? If yes, what are they?

$\boxed{\text{yes, } C/x}$

5. Find an explicit solution of:

$$\frac{dy}{dx} + \frac{y}{\cos^2(3x)} = 0.$$

+10

Be sure to clearly label steps to maximize your score.

SOL 1 (Linear 1st order method)

Standard Form: Done ✓

Integrating Factor:

$$e^{\int P(x)dx} = e^{\int \frac{1}{\cos^2(3x)}dx}$$

$$= e^{\int \sec^2(3x)dx}$$

$$= e^{\frac{1}{3}\tan(3x)} \quad +5$$

Multiply:

$$e^{1/3\tan(3x)}\left(\frac{dy}{dx} + \frac{y}{\cos^2(3x)}\right) = 0$$

$$\frac{d}{dx}\left[e^{1/3\tan(3x)} \cdot y\right] = 0 \quad +2$$

Integrate:

$$e^{1/3\tan(3x)} \cdot y = C \quad +2$$

$$y = \frac{C}{e^{1/3\tan(3x)}} \quad +1$$

OR

SOL 2 (separable equation.)

$$\frac{dy}{dx} = \frac{-y}{\cos^2(3x)}$$

$$\int \frac{dy}{-y} = \int \frac{dx}{\cos^2(3x)}$$

$$-\ln|y| = \int \sec^2(3x)\,dx$$

$$-\ln|y| = \frac{1}{3}\tan(3x) + E \quad +6$$

$$\ln|y| = -\frac{1}{3}\tan(3x) + D$$

$$e^{\ln|y|} = e^{-\frac{1}{3}\tan(3x)+D} = e^{-1/3\tan(3x)}\underbrace{e^{D}}_{G}$$

$$|y| = Ge^{-1/3\tan(3x)}$$

$$y = \pm He^{-1/3\tan(3x)}$$

$$y = Ce^{-1/3\tan(3x)} \quad +4$$

Explicit Solution: $\boxed{y = Ce^{-1/3\tan(3x)}}$

6. (a) What substitution turns the Bernoulli equation $x\frac{dy}{dx}+y=x^2y^2$ into a 1st order linear differential equation?

+2

$$u = y^{1-2} = y^{-1}$$

(b) What substitution turns the homogeneous of degree equation $(xy+y^2)dx + x^2dy = 0$ into a separable differential equation?

+2

$$x = uy \quad \text{or} \quad y = ux$$

(c) Pick one of the two differential equations above to fully solve.

I will solve the differential equation from (a) (b) (CIRCLE ONE)

SOLVING:

+10

(a) $u = y^{-1} = \frac{1}{y}$

$y = \frac{1}{u}$

$\frac{dy}{dx} = -\frac{1}{u^2}\cdot\frac{du}{dx}$

$x\left[\frac{-1}{u^2}\frac{du}{dx}\right] + \left[\frac{1}{u}\right] = x^2\left[\frac{1}{u^2}\right]$

$\frac{-1}{u^2}\frac{du}{dx} + \frac{1}{x}\frac{1}{u} = x\frac{1}{u^2}$

$\frac{du}{dx} - \frac{1}{x}u = -x$ +4

1st order linear now

$e^{\int P(x)dx} = e^{\int -\frac{1}{x}dx} = e^{-\ln|x|}$

$= e^{\ln(1/|x|)}$

$= \frac{1}{x}$ +3

Multiply:

$\frac{1}{x}\left(\frac{du}{dx} - \frac{1}{x}u\right) = \frac{1}{x}(-x)$

$\frac{d}{dx}\left[\frac{1}{x}\cdot u\right] = -1$ +1

integrate:

$\frac{u}{x} = -x + C$

$u = -x^2 + Cx$

$\frac{1}{y} = -x^2 + Cx$ +1

$$\boxed{y = \frac{1}{-x^2+Cx}}$$ +1

SOLVING

(b) $x = uy$

$dx = u\,dy + y\,du$

$(xy+y^2)dx + x^2dy = 0$

$(uy^2 + y^2)(u\,dy + y\,du) + (uy)^2dy = 0$

$u^2y^2dy + uy^2dy + uy^3du + y^3du + u^2y^2dy = 0$

$u^2y^2dy + uy^2dy + u^2y^2dy = -uy^3du - y^3du$

$(u^2y^2 + uy^2 + u^2y^2)dy = (-uy^3 - y^3)du$

$(2u^2y^2 + uy^2)dy = -y^3(u+1)du$

$y^2u(2u+1)dy = -y^3(u+1)du$

$\frac{y^2}{y^3}dy = -\frac{u+1}{u(2u+1)}du$ +5

$\int\frac{1}{y}dy = -\int\frac{u+1}{u(2u+1)}du$

$\frac{u+1}{u(2u+1)} = \frac{A}{u} + \frac{B}{2u+1} = \frac{A(2u+1)+Bu}{u(2u+1)}$

$u+1 = 2Au + A + Bu$

$1 = 2A + B$ and $1 = A$

$1 = 2 + B$

$B = -1$

$\ln|y| = -\int\left(\frac{1}{u} - \frac{1}{2u+1}\right)du = -\ln|u| + \frac{1}{2}\ln|2u+1|$ +4

$$\boxed{\ln|y| = -\ln\left|\frac{x}{y}\right| + \frac{1}{2}\ln\left|2\left(\frac{x}{y}\right)+1\right|}$$ +1

Implicit (or Explicit) Solution: