TEST 2

Math 271 - Differential Equations

Score: ______ out of 100

3/19/2014

Name: Key

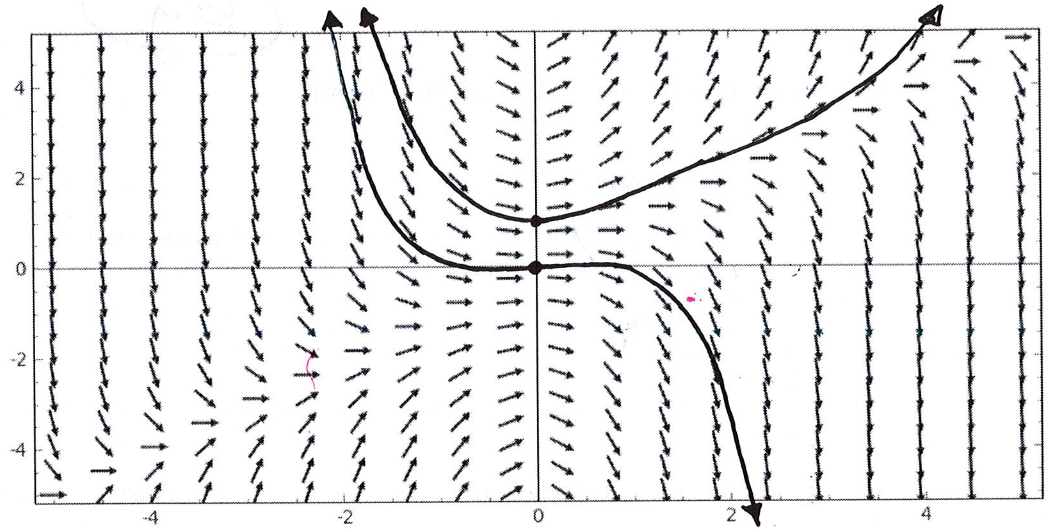
Read all of the following information before starting the exam:

- You have 50 minutes to complete the exam.
- Show all work, clearly and in order, if you want to get full credit. Please make sure you read the directions for each problem. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Please box/circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements.
- This test has 7 problems and is worth 100 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

1. The following is the direction field for the differential equation

$$\frac{dy}{dx} = xy - x^2,$$

over the region $R = \{(x, y) \mid -5 \leq x \leq 5, -5 \leq y \leq 5\}$.



Sketch an approximate solution curve that passes through the following points:

(a) $y(0) = 0$.

(b) $y(0) = 1$

Use your solution curve that passes through the point $y(0) = 0$ to estimate the value of $y(-2)$.

$y(-2) =$ 4 (approximately. something between 1 and 5 really)

2. The function $y_1 = \ln(x)$ is a solution to $xy'' + y' = 0$. Use the reduction of order equation formula to find a second solution $y_2(x)$. (NOTE: you do not need to vertify that y_1 is a solution, just find y_2.)

Standard Form: $y'' + \underbrace{\frac{1}{x}}_{P(x)} y' = 0$ +2

$$y_2 = y_1 \int \frac{e^{-\int P(x)dx}}{(y_1)^2} dx = \ln x \int \frac{e^{-\int 1/x \, dx}}{(\ln(x))^2} dx$$ +2

$$= \ln x \int \frac{e^{-\ln x}}{(\ln x)^2} dx$$ +2

$$= \ln x \int \frac{e^{\ln(\frac{1}{x})}}{(\ln x)^2} dx$$ +2

$$= \ln x \int \frac{1}{x(\ln x)^2} dx$$ + 2

$u = \ln x \rightarrow \frac{du}{dx} = \frac{1}{x} \rightarrow x\,du = dx$ +5

$$= \ln x \int \frac{1}{xu^2} \cdot x\,du = \ln x \int \frac{1}{u^2} dx = \ln x \left(\frac{-1}{u}\right)$$

$$= \ln x \left(-\frac{1}{\ln x}\right) = \boxed{-1}$$

3. Determine whether the given set of functions is linearly independent on the interval $(0, \infty)$. SHOW WORK AND CLEARLY STATE whether the set of functions is **linearly independent** or **linearly dependent.**

(a) $f_1(x) = e^{2x}$, $f_2(x) = e^{3x}$

+2 +1

$$\begin{vmatrix} e^{2x} & e^{3x} \\ 2e^{2x} & 3e^{3x} \end{vmatrix} = 3e^{2x}e^{3x} - 2e^{2x}e^{3x}$$

+1

$$= e^{2x}e^{3x} = e^{5x} \neq 0 \text{ on } (0,\infty).$$

$\boxed{\text{linealy independent.}}$ +1

(b) $g_1(x) = -\sin^2(x)$, $g_2(x) = 2\cos^2(x)$, $g_3(x) = 3$

+5

SOL 1: $(-1)(\sin^2(x)) + (\tfrac{1}{2})(2\cos^2(x)) + (-\tfrac{1}{3})(3) = 0$ $\boxed{\text{linealy dependent}}$

+2

SOL 2: $\begin{vmatrix} -\sin^2(x) & 2\cos^2(x) & 3 \\ -2\sin(x)\cos(x) = -\sin(2x) & -4\cos(x)\sin(x) = -2\sin(2x) & 0 \\ -2\cos(2x) & -4\cos(2x) & 0 \end{vmatrix} = -\sin^2(x)(0) - 2\cos^2(x)(0) + 3\begin{vmatrix} -\sin(2x) & -2\sin(2x) \\ -2\cos(2x) & -4\cos(2x) \end{vmatrix}$

+1

$= 3(+4\sin(2x)\cos(2x) - 4\sin(2x)\cos(2x))$

$= 0 \rightarrow \boxed{\text{linealy dependent}}$

+1 +1

4. Complete all of the following parts. **You may not use the auxiliary/characteristic equation method!**

(a) Verify that $y_1 = x$ and $y_2 = x\ln(x)$ form a fundamental set of solutions of $x^2y'' - xy' + y = 0$ on $(0, \infty)$.

$y_1 = x$

$y_1' = 1$

$y_1'' = 0$

$y_2 = x\ln x$

$y_2' = x(\tfrac{1}{x}) + \ln x = 1 + \ln x$

$y_2'' = \tfrac{1}{x}$

$$\begin{vmatrix} x & x\ln x \\ 1 & 1+\ln x \end{vmatrix} = x + x\ln x - x\ln x = x \neq 0 \text{ on } (0,\infty)$$

$\boxed{\text{linealy independent}}$

LHS $= x^2y'' - xy' + y$
$= x^2 \cdot 0 - x \cdot 1 + x$
$= 0$ ✓
$=$ RHS

LHS $= x^2y'' - xy' + y$
$= x^2(\tfrac{1}{x}) - x(1+\ln x) + x\ln x$
$= x - x - x\ln x + x\ln x$
$= 0 =$ RHS ✓

(b) Verify that $y_p = 2 + \ln(x)$ forms a particular solution of $x^2y'' - xy' + y = \ln(x)$.

$y_p = 2 + \ln(x)$

$y_p' = \tfrac{1}{x}$

$y_p'' = -\tfrac{1}{x^2}$

$\rightarrow$ $x^2y'' - xy' + y =$

$x^2(-\tfrac{1}{x^2}) - x(\tfrac{1}{x}) + 2 + \ln(x) =$

$-1 - 1 + 2 + \ln(x) = \ln(x)$ ✓

(c) Use (a) and (b) to write the general solution of $x^2y'' - xy' + y = \ln(x)$.

General Solution: $\boxed{y = \underbrace{C_1x + C_2x\ln x}_{y_c} + \underbrace{2 + \ln(x)}_{y_p}}$

5. Find the general solution to the following:

(a) $y'' - 4y' + 5y = 0$

$$m^2 - 4m + 5 = 0$$

$$m = \frac{4 \pm \sqrt{(-4)^2 - 4(5)}}{2} = \frac{4 \pm \sqrt{16-20}}{2} = \frac{4 \pm \sqrt{-4}}{2}$$

$$= \frac{4 \pm 2i}{2}$$

$$= 2 \pm i$$

$$\alpha = 2, \quad \beta = 1$$

$$\boxed{y = C_1 e^{2x} \cos(x) + C_2 e^{2x} \sin(x)}$$

(b) $y''' + 2y'' - 4y' - 8y = 0$

$$m^3 + 2m^2 - 4m - 8 = 0$$

$$m^2(m+2) - 4(m+2) = 0$$

$$(m+2)(m^2 - 4) = 0$$

$$(m+2)(m+2)(m-2) = 0$$

$$m = -2 \quad \Big| \quad m = -2 \quad \Big| \quad m = 2$$

$$\boxed{y = C_1 e^{2x} + C_2 e^{-2x} + C_3 x e^{-2x}}$$

(c) $y^{(6)} - 9y^{(4)} = 0$

$$m^6 - 9m^4 = 0$$

$$m^4(m^2 - 9) = 0$$

$$m^4(m-3)(m+3) = 0$$

$$\underbrace{m = 0}_{\text{4 times}} \quad \Big| \quad m = 3 \quad \Big| \quad m = -3$$

(multiplicity 4)

$$\boxed{y = C_1 + C_2 x + C_3 x^2 + C_4 x^3 + C_5 e^{3x} + C_6 e^{-3x}}$$

6. Solve the following differential equation using the method of undetermined coefficients:

$$y'' + 3y' + 2y = 4x^2$$

Find y_c: $m^2 + 3m + 2 = 0$

$(m+2)(m+1) = 0$

$m = -2 \quad | \quad m = -1$

$y_c = C_1 e^{-2x} + C_2 e^{-x}$ +5

Find y_p: Form from $g(x) = 4x^2$ +5

$y_p = Ax^2 + Bx + C$

$y_p' = 2Ax + B$ +2

$y_p'' = 2A$

$y'' + 3y' + 2y = 4x^2$ becomes

$2A + 3(2Ax + B) + 2(Ax^2 + Bx + C) = 4x^2$

$2A + 6Ax + 3B + 2Ax^2 + 2Bx + 2C = 4x^2$ +2

$2A = 4$

$A = 2$

$6A + 2B = 0$

$12 + 2B = 0$

$2B = -12$

$B = -6$

$2A + 3B + 2C = 0$

$4 - 18 + 2C = 0$

$-14 + 2C = 0$

$2C = 14$

$C = 7$

+5

$y_p = 2x^2 - 6x + 7$

$y = y_c + y_p$

General Solution: $\boxed{y = C_1 e^{-2x} + C_2 e^{-x} + 2x^2 - 6x + 7}$ +1

7. Solve the following differential equation using the <u>variation of parameters</u>:

$$y'' - 4y' + 4y = (x+1)e^{2x}$$

Hint: make sure you simplify the Wronskian!

Standard Form: DONE! $f(x) = (x+1)e^{2x}$

Find y_c: $y'' - 4y' + 4y = 0$

$$m^2 - 4m + 4 = 0$$

$$(m-2)(m-2) = 0$$

$$m = 2 \;|\; m = 2$$

$$y = C_1 \underbrace{e^{2x}}_{y_1} + C_2 \underbrace{xe^{2x}}_{y_2}$$

+5

Find y_p: $y_p = u_1 y_1 + u_2 y_2$ +2

$$W(e^{2x}, xe^{2x}) = \begin{vmatrix} e^{2x} & xe^{2x} \\ 2e^{2x} & x(2e^{2x}) + e^{2x} \end{vmatrix} = e^{2x}(2xe^{2x} + e^{2x}) - 2xe^{2x}e^{2x}$$

$$= \cancel{2xe^{4x}} + e^{4x} - \cancel{2xe^{4x}}$$

$$= e^{4x}$$

+4

$$u_1 = \int \frac{-y_2 f(x)}{W}\,dx = \int \frac{-x\cancel{e^{2x}}(x+1)\cancel{e^{2x}}}{\cancel{e^{4x}}}\,dx = \int -x(x+1)\,dx = -\int (+x^2 + x)\,dx$$

$$= -\left(+\frac{x^3}{3} + \frac{x^2}{2}\right)$$

+4

$$u_2 = \int \frac{y_1 f(x)}{W}\,dx = \int \frac{\cancel{e^{2x}}(x+1)\cancel{e^{2x}}}{\cancel{e^{4x}}}\,dx = \int (x+1)\,dx = \frac{x^2}{2} + x$$

+4

$$y_p = -\left(+\frac{x^3}{3} + \frac{x^2}{2}\right)e^{2x} + \left(\frac{x^2}{2} + x\right)xe^{2x}$$

$$y = y_c + y_p$$

General Solution: $y = \boxed{C_1 e^{2x} + C_2 xe^{2x} - \left(+\frac{x^3}{3} + \frac{x^2}{2}\right)e^{2x} + \left(\frac{x^2}{2} + x\right)xe^{2x}}$ +1