TEST 3

Math 271 - Differential Equations **Score:** ________ out of 100

4/23/2014 **Name:** Key

Read all of the following information before starting the exam:

- You have 50 minutes to complete the exam.
- Show all work, clearly and in order, if you want to get full credit. Please make sure you read the directions for each problem. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Please box/circle or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements.
- This test has 5 problems and is worth 100 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

1. $x = 0$ is an ordinary point of the differential equation:

$$y'' - xy' + 2y = 0.$$

Find two linearly independent power series solutions about $x = 0$. You should write down the first three nonzero terms of each series solution. (if possible).

$$y = \sum_{n=0}^{\infty} c_n x^n$$

$$y' = \sum_{n=1}^{\infty} c_n \cdot n x^{n-1}$$

$$y'' = \sum_{n=2}^{\infty} c_n \cdot n(n-1) x^{n-2}$$

procedure:
0) bring in coefficients
1) Fix phases
2) Fix index (shift)

(+5) so $y'' - xy' + 2y = 0$ becomes

$$\sum_{n=2}^{\infty} c_n \cdot n(n-1) x^{n-2} - x \sum_{n=1}^{\infty} c_n \cdot n x^{n-1} + 2 \sum_{n=0}^{\infty} c_n \cdot x^n = 0$$

bring in coef.

$$\sum_{n=2}^{\infty} c_n \cdot n(n-1) x^{n-2} - \sum_{n=1}^{\infty} c_n \cdot n x^n + \sum_{n=0}^{\infty} 2c_n x^n = 0$$

starts x^0 starts x^1 starts x^0

$$2c_2 + \sum_{n=3}^{\infty} c_n \cdot n(n-1) x^{n-2} - \sum_{n=1}^{\infty} c_n \cdot n x^n + 2c_0 + \sum_{n=1}^{\infty} 2c_n x^n = 0$$

fix index

$$(2c_0 + 2c_2) + \sum_{n=1}^{\infty} c_{n+2}(n+2)(n+1) x^n - \sum_{n=1}^{\infty} c_n \cdot n x^n + \sum_{n=1}^{\infty} 2c_n x^n = 0$$

(+5) $$(2c_0 + 2c_2) + \sum_{n=1}^{\infty} \left[c_{n+2}(n+2)(n+1) - c_n \cdot n + 2c_n \right] x^n = 0$$

By the identity property:

$2c_0 + 2c_2 = 0$ AND $c_{n+2}(n+2)(n+1) - c_n \cdot n + 2c_n = 0$ for $n = 1, 2, 3, \ldots$

$c_2 = -c_0$ AND $c_{n+2} = \dfrac{c_n \cdot n - 2c_n}{(n+2)(n+1)}$ for $n = 1, 2, 3, \ldots$

$$c_{n+2} = \frac{(n-2)c_n}{(n+2)(n+1)} \quad \text{for } n = 1, 2, 3, \ldots$$

$c_0 = ?$
$c_1 = ?$
$c_2 = -c_0$

n	$c_{n+2} = \frac{(n-2)c_n}{(n+2)(n+1)}$
1	$c_3 = \frac{(-1)c_1}{(3)(2)} = \frac{-c_1}{3!}$
2	$c_4 = 0$
3	$c_5 = \frac{(-1)c_3}{(5)(4)} = \frac{-1\cdot c_1}{5!}$
4	$c_6 = 0$
	$\vdots$

+5

[redacted]

+5

$$c_0 + c_1 x + (-c_0)x^2 + \left(\frac{-c_1}{3!}\right)x^3 + 0 + \left(\frac{-1}{5!}\right)c_1 x^5$$

$$c_0\underbrace{\left(1 - x^2\right)}_{y_1} + c_1\underbrace{\left(x - \frac{1}{3!}x^3 - \frac{1}{5!}x^5 + \dots\right)}_{y_2}$$

$y_1 =$ $1 - x^2$

$y_2 =$ $x - \frac{1}{3!}x^3 - \frac{1}{5!}x^5 + \dots$

2. Find the following Laplace transforms

10 (a) $\mathscr{L}\{2 + t^5 + e^{-3t}\}$

$$\frac{2}{s} + \frac{5!}{s^6} + \frac{1}{s-(-3)}$$

$$\boxed{\frac{2}{s} + \frac{5!}{s^6} + \frac{1}{s+3}}$$

(b) $\mathscr{L}\{e^{6t}\cos(3t)\}$

10

$$\mathscr{L}\{\cos(3t)\}\Big|_{s \to s-6}$$

$$= \frac{s}{s^2+9}\Bigg|_{s \to s-6}$$

$$\boxed{\frac{s-6}{(s-6)^2+9}}$$

10 (c) $\mathscr{L}\{4t\mathscr{U}(t-9)\}$

$a = 9$

$f(t) = 4t$

$f(t+9) = 4(t+9) = 4t + 36$

$$e^{-9s}\,\mathscr{L}\{4t+36\}$$

$$\boxed{e^{-9s}\left(\frac{4}{s^2} + \frac{36}{s}\right)}$$

3. Find the following **inverse** Laplace transforms

10 (a) $\mathscr{L}^{-1}\left\{\frac{1}{s^5}+\frac{s}{s^2+100}\right\}$

$\frac{1}{4!}\mathscr{L}^{-1}\left\{\frac{4!}{s^5}\right\} + \cos(10t)$

$\boxed{\frac{1}{4!}t^4 + \cos(10t)}$

$\frac{1}{24}$

10 (b) $\mathscr{L}^{-1}\left\{\frac{1}{(s-4)^2+1}\right\}$

$\mathscr{L}^{-1}\left\{\frac{1}{s^2+1}\bigg|_{s\to s-4}\right\}$

$a=4$

$\boxed{\sin(t)e^{4t}}$

10 (c) $\mathscr{L}^{-1}\left\{e^{-5s}\left(\frac{6}{s^2+36}\right)\right\}$

$a=5$ $F(s)$

$f(t) = \sin(6t)$

$f(t-5) = \sin(6(t-5)) = \sin(6t-30)$

$\boxed{\sin(6t-30)\,\mathcal{U}(t-5)}$

OR

$\boxed{\sin(6(t-5))\,\mathcal{U}(t-5)}$

5 4. Write $f(t)$ in terms of unit step functions (Heaviside functions) if

$$f(t) = \begin{cases} 1, & 0 \le t < \pi, \\ \ln(t), & \pi \le t. \end{cases}$$

$f(t) =$ $1 - 1\mathcal{U}(t-\pi) + \ln(t)\mathcal{U}(t-\pi)$

15 5. Use the Laplace transform to solve the following initial value problem:

$$y'' + 9y = 10e^t, \quad y(0) = 0, \quad y'(0) = 0$$

$$\mathcal{L}\{y''\} + 9\mathcal{L}\{y\} = 10\mathcal{L}\{e^t\}$$

$$s^2Y(s) - sy(0) - y'(0) + 9Y(s) = 10\left(\frac{1}{s-1}\right)$$

+5

$$s^2Y(s) + 9Y(s) = \frac{10}{s-1}$$

$$(s^2+9)Y(s) = \frac{10}{s-1}$$

$$Y(s) = \frac{10}{(s^2+9)(s-1)} = \frac{As+B}{s^2+9} + \frac{C}{s-1}$$

$$10 = (As+B)(s-1) + C(s^2+9)$$

$$10 = As^2 + Bs - As - B + Cs^2 + 9C$$

+5

$10 = -B + 9C$ | $A + C = 0$ | $B - A = 0$

$10 = C + 9C$ | $B = -C$ | $A = B$

$10 = 10C$

$1 = C$ | $B = -1$ | $A = -1$

$$y = \mathcal{L}^{-1}\left\{\frac{-s}{s^2+9}\right\} + \mathcal{L}^{-1}\left\{\frac{-1}{s^2+9}\right\} + \mathcal{L}\left\{\frac{1}{s-1}\right\}$$

+5

$y(t) =$ $-\cos(3t) - \frac{1}{3}\sin(3t) + e^{t}$