COMPLEX ANALYSIS PROBLEMS (MATH 400) SPRING 2013 UPDATED ON April 23, 2013

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Note: ♠ problems are optional challenge problems.

(1) Compute and simplify to $z = a + bi$ form
  (a) i^3
  (b) i^{2013}
  (c) $(3 - 2i) + (-1 + 6i)$
  (d) $(1 + i) - (2 + \sqrt{2}i)$
  (e) $(-1 + i)(4 - i)$
  (f) $(-1 + i)^3$
  (g) $(2 - i)(\overline{1 + i})$
  (h) $\mathrm{Re}[(-1 + i)^3]$
  (i) $\mathrm{Im}[(-1 + i)^3]$
  (j) $i/(1 - i)$
  (k) $(2 - i)/(\overline{1 + i})$

(2) Let $z_1 = 1 + 3i$, $z_2 = -1 - 2i$.
  (a) In the $\mathbb{C}$omplex plane, plot z_1, z_2, $z_1 + z_2$, $z_1 - z_2$, $z_1 z_2$ and z_1/z_2.
  (b) Compute the following:
    (i) $|z_1|$
    (ii) $|z_2|$
    (iii) $|z_1 z_2|$
    (iv) $|z_1 - z_2|$ (do this two different ways)

(3) Compute and simplify to $z = a + bi$ form
  (a) $\left|\dfrac{1}{1+i}\right|$
  (b) $|(2 - i\sqrt{2})^{50}|$

(4) Describe and graph the set of points $z \in \mathbb{C}$ that satisfy the following.
  (a) $z = \overline{z}$
  (b) $-z = \overline{z}$
  (c) $\overline{z} = z^{-1}$
  (d) $|z - 2 + i| = 3$
  (e) $\mathrm{Im}((1 - i)z + 2) = 0$
  (f) $|z - i| = |z + 1|$
  (g) $|\overline{z}| = \mathrm{Im}(z)$
  (h) $\mathrm{Re}(z\overline{z}) = 4$

(5) Compute the following.
   (a) $\operatorname{Arg}(1)$
   (b) $\arg(1)$
   (c) $\operatorname{Arg}(3i)$
   (d) $\arg(3i)$
   (e) $\operatorname{Arg}(1+\sqrt{3}i)$
   (f) $\arg(1+\sqrt{3}i)$
   (g) $\operatorname{Arg}((1-i)^5)$

(6) Write the following complex numbers in $z = a + bi$ form
   (a) $2\left[\cos\left(\frac{11\pi}{3}\right) + i\sin\left(\frac{11\pi}{3}\right)\right]$
   (b) $\sqrt{5}\left[\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right]$
   (c) $10\left[\cos\left(\frac{-\pi}{3}\right) + i\sin\left(\frac{-\pi}{3}\right)\right]$

(7) Write the following complex numbers in polar form ($z = r[\cos(\theta) + i\sin(\theta)]$.
   (a) $\sqrt{3} - i$
   (b) $1 + \sqrt{3}i$
   (c) $2 - \sqrt{3} + i$

(8) Use your answers from (7) to help you write the following in polar form.
   (a) $(\sqrt{3} - i)^{100}$
   (b) $(1 + \sqrt{3}i)^{2012}$
   (c) $(2 - \sqrt{3} + i)^{216}$

(9) ♠ Prove: For all $n \geq 1$, For all $z_1, z_2, \ldots, z_n \in \mathbb{C}$, $\left|\sum_{j=1}^{n} z_j\right| \leq \sum_{j=1}^{n} |z_j|$.

(10) Find and graph all roots in both $a + bi$ and polar forms.
   (a) $(1)^{(1/5)}$
   (b) $(-1)^{(1/4)}$
   (c) $(-1+i)^{(1/3)}$
   (d) $(4i)^{(1/6)}$

(11) The n**th roots of unity** are the n distint solutions to the equation $z^n = 1$.
   (a) Show that the nth roots of unity are given by

$$(1)^{(1/n)} = \omega_k = \cos\left(\frac{2\pi k}{n}\right) + i\sin\left(\frac{2\pi k}{n}\right),$$

   where $k = 0, 1, \ldots n - 1$. Hint: Use your answer from (10a) to help.

(b) ♠ Let $n > 1$ be fixed. For which k can we obtain all of the nth roots of unity by taking powers of ω_k? Hint: this should work for $k = 1$: $\omega_1, \omega_1^2, \omega_1^3, \ldots$ should give every ω_k, but are there other values of k that work as well? What is the relationship between k and n here?

(c) ♠ Let $n > 1$ be fixed. For the values of k where ω_k does not generate all of the roots of unity, how many of the roots are found by taking powers of ω_k? Can you come up with a way to relate k and n and the number of generated roots? Draw some pictures and try this for various k and at least $n = 2, 3, 4, 5, 6$, and then think of the general situation.

(d) ♠ Prove that $1 + \omega_1 + \omega_1^2 + \cdots + \omega_1^{n-1} = 0$. Hint: Use $\omega_1^n = 1$.

(12) Compute $f(1)$, $f(i)$ and $f(1+i)$ for the following functions.
   (a) $f(z) = z + (\overline{z})^{-1}$.
   (b) $f(z) = z^2 - \mathrm{Re}(z)$
   (c) $f(z) = \mathrm{Re}(z) - \mathrm{Im}(z)$
   (d) $f(z) = e^z$
   (e) $f(z) = 3x - iy\overline{z}$

(13) Find the real and imaginary parts of the following functions and rewrite the functions in the form $f(z) = u(x, y) + iv(x, y)$.
   (a) $f(z) = z^2$
   (b) $f(z) = z + z^{-1}$
   (c) $f(z) = e^{\overline{z}}$
   (d) $f(z) = iz - \mathrm{Im}(z)|z|$

(14) (a) Let $z = x + iy \in \mathbb{C}$. Show that $|e^z| = e^x$.
   (b) ♠ Let $z_1, z_2 \in \mathbb{C}$. Show that $e^{z_1+z_2} = e^{z_1}e^{z_2}$.

(15) Find the image of the following sets under the given complex function. Draw a picture of the set and its image (z-plane and w-plane pictures).
   (a) $w = f(z) = -iz$ and the set of $z \in \mathbb{C}$ such that $\mathrm{Im}(z) = -5$. Note: this is the same function from class.
   (b) $w = f(z) = -iz$ and the set of $z \in \mathbb{C}$ such that $\mathrm{Im}(z) = 5$. Note: this is the same function from class.
   (c) $w = f(z) = -iz$ and the set of $z \in \mathbb{C}$ such that $-5 < \mathrm{Im}(z) < 5$. Note: this is the same function from class.
   (d) $w = f(z) = z + 3i$ and the set of $z \in \mathbb{C}$ such that $x = y$.
   (e) $w = f(z) = z + 3i$ and the set of $z \in \mathbb{C}$ such that $|z - 1| < 1$.
   (f) $w = f(z) = iz - \overline{z}$ and the set of $z \in \mathbb{C}$ such that $\mathrm{Re}(z) + \mathrm{Im}(z) \geq 1$.

(16) Find the parametrization of the following curves:
   (a) The line through 8 and $1 + i$.

(b) The line segment connecting -1 and $2-i$.
(c) The ray starting at $-6i$ and going through $1+3i$.
(d) The circle of radius 4 centered at $2i$ oriented positively.

(17) Find the image of the following curves under the given mapping (Hint: use the parametrizations from the previous problem).
(a) The line through 8 and $1+i$ under $f(z) = e^{i\pi/3}z$.
(b) The line segment connecting -1 and $2-i$ under $f(z) = e^{i\pi/4}z - 3i + 1$.
(c) The ray starting at $-6i$ and going through $1+3i$ under $f(z) = -z$.
(d) The circle of radius 4 centered at $2i$ oriented positively under $f(z) = 3z + i - 1$.

(18) Classify the following functions as translations, rotations, magnifications or other?
(a) $f(z) = e^{i\pi/3}z$.
(b) $f(z) = \overline{z}$.
(c) $f(z) = \operatorname{Im}(z) + \operatorname{Re}(z)$.
(d) $f(z) = -z$.
(e) $f(z) = \sqrt{2}z$.
(f) $f(z) = 3z + i - 1$.

(19) Show that the following functions $f : \mathbb{C} \to \mathbb{C}$ are NOT one-to-one (injective)
(a) $f(z) = z^2$.
(b) $f(z) = z^5$.

(20) Show that the following functions $f : \mathbb{C} \to \mathbb{C}$ are one-to-one (injective), and then find f^{-1}.
(a) $f(z) = z - i\sqrt{3}$.
(b) $f(z) = 4iz - 2$.

(21) Show that the following limits do not exist.
(a) $\displaystyle\lim_{z\to 0} \frac{\operatorname{Re}(z)}{z}$
(b) $\displaystyle\lim_{z\to 0} \frac{\operatorname{Im}(z)^2 + 2}{3z^2}$

(22) Evaluate $\lim_{z\to z_0} f(z)$ by writing $f(z) = u(x,y) + iv(x,y)$ and computing the real limits.
(a) $\displaystyle\lim_{z\to 1-i} \left(\frac{x}{x^2 + y^2 + 1} + 2xyi \right)$
(b) $\displaystyle\lim_{z\to i} \left(i|z|^2 - \operatorname{Re}(z)\operatorname{Im}(z) \right)$
(c) $\displaystyle\lim_{z\to 0} \frac{|z|^2}{\overline{z}}$

(d) $\lim_{z \to z_0} \operatorname{Re}(z)$
(e) $\lim_{z \to z_0} \operatorname{Im}(z)$
(f) ♠ Show: $\lim_{z \to z_0} c = c$, where $c \in \mathbb{C}$ is a constant.
(g) ♠ Show: $\lim_{z \to z_0} z = z_0$.

(23) Show that the following functions are continuous at the given point.
(a) $f(z) = z^2 - 4iz + 3$; $z_0 = 1 - i$.
(b) $f(z) = z - \frac{1}{z}$; $z_0 = -2i$.
(c) $f(z) = \frac{z^2}{1-z}$; $z_0 = -i$.

(24) Show that the following functions are discontinuous at the given point.
(a) $f(z) = z - \frac{1}{z}$; $z_0 = 0$.
(b) $f(z) = \frac{z^2}{1+z^2}$; $z_0 = i$.

(25) Determine where the following functions are continuous
(a) $f(z) = \frac{z-2}{z^3+i}$
(b) $f(z) = \frac{z}{x-1}$
(c) $f(z) = \frac{2iz^2}{|z|+1}$
(d) $f(z) = \frac{2iz^2}{|z|-1}$

(26) Compute $f'(z)$ using the limit definition of the derivative
(a) $f(z) = 2i$
(b) $f(z) = 2i/z$

(27) Compute $f'(z)$ using the derivative formulas
(a) $f(z) = 2i$
(b) $f(z) = 2i/z$
(c) $f(z) = (z^3 - 2\pi i)^{2013}$

(28) Show that the following functions are nowhere differentiable
(a) $f(z) = \overline{z}$
(b) $f(z) = 2ix + 4y$
(c) ♠ $f(z) = |z|$

(29) Use the Cauchy-Riemann equations to help determine where the following functions are differentiable and evaluate the derivative (valid at those points where they exist).
(a) $f(z) = 2iz + 4 - i$
(b) $f(z) = x^2 + iy^2$

(c) $f(z) = 3x^3 + (y^2 + 2y)i$
(d) $f(z) = -\cos(x) + y\sin(x)i$
(e) $f(z) = |z + 1 - 2i|^2$
(f) $f(z) = \sin(x)\cosh(y) + i\cos(x)\sinh(y)$

(30) ♠ Suppose $f(z)$ and $\overline{f(z)}$ are both analaytic on some domain D. Show that $f(z)$ must be constant on D.

(31) Which of the following functions are harmonic?
(a) $h(x, y) = x^2 - y^2$
(b) $h(x, y) = \sin(x)e^y$
(c) $h(x, y) = x^3 - y$

(32) Compute
(a) $\ln(-1 - i)$
(b) $\mathrm{Ln}(-1 - i)$
(c) $\ln(1 - \sqrt{3}i)$
(d) $\mathrm{Ln}(1 - \sqrt{3}i)$
(e) $\ln(5)$
(f) $\mathrm{Ln}(5)$

(33) Find all $z \in \mathbb{C}$ that satisfy the given equation
(a) $e^z = -i\pi$
(b) $\mathrm{Ln}(z + 1) = 2\pi i$
(c) $e^{z^3} = 2i$

(34) Let $p(z) = a_n z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$, where $a_0, a_1, \ldots, a_n$ are real constants (a complex polynomial with real coefficients). Show that if $p(z) = 0$, then $p(\overline{z}) = 0$.

(35) Differentiate the following
(a) $\mathrm{Ln}(z^3 + 1)$
(b) $e^{4\pi i z}$
(c) $\sin(e^z)$
(d) $\cos(z^2 + \cosh(z))$

(36) Find the principal value of the following
(a) i^{2i}
(b) $(1 - i)^i$

(37) Find all $z \in \mathbb{C}$ that satisfy the given equation
(a) $\cos(z) = i$

(38) Use a parametrization of C to evaluate the following
(a) $\displaystyle\int_C x\,dz$, where C is the line segment from 0 to $2i$.
(b) $\displaystyle\int_C x\,dz$, where C is right half of the circle $|z - i| = 1$, oriented counterclockwise.

(c) $\oint_C x dz$, where C is $|z-i|=1$.

(d) $\int_C y dz$, where C is the line segment from 0 to $2i$.

(e) $\oint_C \frac{1}{z} dz$, where C is $|z|=10$.

(39) Use the Cauchy-Goursat Theorem, Annulus Theorem (Deformation of Contour Theorem), and Generalized Deformation of Contours Theorem (Cauchy-Goursat Theorem for Multiply Connected Domains) to evaluate the following

(a) $\oint_C \frac{z-1}{z} dz$, where C is $|z-i|=1/2$.

(b) $\oint_C \frac{z-1}{z} dz$, where C is $|z-i|=5$.

(c) $\oint_C \frac{1}{z^2+1} dz$, where C is $|z|=1/2$.

(d) $\oint_C \frac{1}{z^2+1} dz$, where C is $|z|=2$.

(e) $\oint_C \frac{1}{z^2+1} dz$, where C is $|z-i|=1/\pi$.

(f) $\oint_C \frac{1}{z^2+i} dz$, where C is $|z|=2$.

(g) $\oint_C e^z dz$, where C is any simple closed curve in the complex plane that traces out the shape of a $\mathbb{C}$lover ♣.

(40) Compute the integral $\int_C (7z - e^z) dz$, where C is upper half of the circle $|z|=1$, oriented clockwise, in the following 3 ways

(a) Parametrize C.

(b) Use a different curve C'.

(c) Use the Fundamental Theorem for Contour Integrals.

(41) Evaluate the following using the Fundamental Theorem for Contour Integrals.

(a) $\int_C e^z dz$, where C is the line segment from -1 to i.

(b) $\int_C \sin(4z) dz$, where C is the line segment from $2i$ to $3i$.

(c) $\int_C \cos^2(2z) dz$, where C is the line segment from 0 to $\pi/2$.

(d) $\int_C \cosh(2z) dz$, where C is the line segment from 0 to πi.

(42) Evaluate the following using Cauchy's integral formula.

(a) $\oint_C \frac{e^z}{z-5}dz$, where C is $|z-1| = 10$.
(b) $\oint_C \frac{1}{z^2+1}dz$, where C is $|z| = 1/2$.
(c) $\oint_C \frac{1}{z^2+1}dz$, where C is $|z| = 2$.
(d) $\oint_C \frac{1}{z^2+1}dz$, where C is $|z-i| = 1/\pi$.
(e) $\oint_C \frac{1}{z^4-1}dz$, where C is $|z| = 2$.
(f) ♠ $\oint_C \frac{1}{z^n-1}dz$, where C is $|z| = 2$.

(43) Evaluate the following using Cauchy's integral formula.
(a) $\oint_C \frac{e^z}{(z-2)^3}dz$, where C is $|z| = 4$.
(b) $\oint_C \frac{1}{z\sin(z)}dz$, where C is $|z| = 1$.
(c) $\oint_C \frac{\cosh(z)}{z^4}dz$, where C is $|z| = 1$.
(d) $\oint_C \frac{1}{z^2(z^2-1)}dz$, where C is $|z| = 2$.

(44) Find the radius of convergence for the following power series
(a) $\sum_{n=0}^{\infty} \frac{z^n}{2n+1}$.
(b) $\sum_{n=0}^{\infty} \frac{(z-2i)^n}{(2n)!}$.

(45) Find the Maclaurin series representation for the following functions and find the radius of convergence of the representation. Hint: use any of the Maclaurin series from class.
(a) e^{z^2}
(b) $\sin(z^2)$
(c) $\frac{1}{1+z^3}$.
(d) $\cosh(z)$.
(e) $\frac{1}{(1-z)^2}$.
(f) $\mathrm{Ln}(1+z)$.

(46) Find the Laurent series representation for the following functions valid on the given annular domain.
(a) $\frac{e^{z^2}}{z^2}$, $|z| > 0$.

(b) $\dfrac{1+z-e^z}{z}$, $|z|>0$.
(c) $\dfrac{\sin(z)-z}{z^3}$, $|z|>0$.
(d) e^{1/z^2}, $|z|>0$.
(e) $\dfrac{1}{z(z+i)}$, $0<|z|<1$.
(f) $\dfrac{1}{z(z+i)}$, $|z|>1$.
(g) $\dfrac{1}{z(z+i)}$, $0<|z+i|<1$.
(h) $\dfrac{1}{z(z+i)}$, $|z+i|>1$.

(47) Determine the zeros and their order for the given functions.
(a) $(2i-z)^5$.
(b) $\cos(z)$
(c) $(z\cos(z))^2$.
(d) $\sin(z)(\cos(z)-1)$.

(48) Classify the singularities for the given functions (state for each singularity if it is removable, simple, a pole of order n, or essential).
(a) $\dfrac{1}{z(z-3i)^4}$.
(b) $\dfrac{1}{(z^2+1)^3}$.
(c) $\dfrac{1}{\cos(z)-1}$
(d) $\dfrac{\sin(z)}{z}$.
(e) $\dfrac{e^{1/z}}{z^{12}}$.
(f) $\dfrac{z-5}{(z-i)^2(z+i)^7}$.
(g) $\dfrac{\sin(z)}{ze^z-z}$

(49) Evaluate the following using Cauchy's Residue Theorem.
(a) $\displaystyle\oint_C \frac{1}{z^2+1}dz$, where C is $|z|=2$.
(b) $\displaystyle\oint_C \frac{\cos(2z)-1}{z^3}dz$, where C is $|z|=1$.
(c) $\displaystyle\oint_C \frac{e^z}{z^5}dz$, where C is $|z|=1$.
(d) $\displaystyle\oint_C \frac{1}{1-e^z}dz$, where C is $|z|=1$.

(e) $\oint_C \frac{1}{z^2(z-3i)}dz$, where C is $|z|=4$.

(f) $\oint_C \frac{1}{z^2(z-3i)}dz$, where C is $|z|=1$.

(g) $\oint_C \frac{1}{z\sin^2(z)}dz$, where C is $|z|=3$.

(h) $\oint_C \cos(2/z)dz$, where C is $|z|=1$.

(i) $\oint_C ze^{\pi/z^2}dz$, where C is $|z|=1$.

(j) $\oint_C \frac{\mathrm{Ln}(z)}{\cos(z)}dz$, where C is $|z-\pi/2|=1/2$.

(k) ♠ $\oint_C \frac{1}{z^n\sin(z)}dz$, where C is $|z|=1$ and $n\in\mathbb{Z}$ with $n\geq 1$.

(50) Evaluate.

(a) $\int_0^{2\pi} \frac{1}{1-\cos(x)}dx$

(b) $\int_{-\infty}^{\infty} \frac{x-1}{x^4+1}dx$

(c) $\int_{-\infty}^{\infty} \frac{\sin(x)}{x^4+1}dx$

(d) $\int_{-\infty}^{\infty} \frac{1}{x(x^2+4)}dx$

(e) $\int_{-\infty}^{\infty} \frac{\cos(x)}{x(x^2+4)}dx$

(f) ♠ $\int_{-\infty}^{\infty} \frac{\cos(x)}{x^n-1}dx$

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