Exam 3A Solutions MTH 201 Fall 2013

1. (a) $\lim\limits_{t\to\infty} \dfrac{e^t+t}{t^2} = \lim\limits_{t\to\infty} \dfrac{e^t+1}{2t} = \lim\limits_{t\to\infty} \dfrac{e^t}{2} = \infty$

   (b) $\lim\limits_{x\to 0^+} x^{\sin x} = \lim\limits_{x\to 0^+} e^{\ln[x^{\sin x}]} = \lim\limits_{x\to 0^+} e^{\sin x \ln x} = e^{\lim_{x\to 0^+} \frac{\ln x}{\csc x}} = e^{\lim_{x\to 0^+} \frac{\frac{1}{x}}{-\csc x \cot x}} = e^{\lim_{x\to 0^+} -\frac{\sin x}{x}\cdot \tan x} = e^0 = 1$

2. Find f given $f'(t) = \sin t - 4e^t + 7$ and $f(0) = 3$.

   The general antiderivative is $f(t) = -\cos t - 4e^t + 7t + c$. Use the initial value to solve for c.

$$\begin{aligned} f(0) &= -\cos 0 - 4e^0 + 7\cdot 0 + c \\ 3 &= -1 - 4 + c \\ 8 &= c \end{aligned}$$

   Thus $f(t) = -\cos t - 4e^t + 7t + 8$

3. A rectangular field bordering a river is to be fenced in on the other three sides. If the area of the field is to be 5000 square feet, find the minimum amount of fencing needed. Be sure to justify your answer completely using calculus.

   Constraint (area): $xy = 5000$ Thus $y = \frac{5000}{x}$.
   Minimize the amount of fencing: $F = x + 2y$.

$$\begin{aligned} F &= x + 2y \\ F(x) &= x + 2\tfrac{5000}{x} \\ F(x) &= x + \tfrac{10{,}000}{x} \\ F'(x) &= 1 - \tfrac{10{,}000}{x^2} \\ F'(x) &= \tfrac{x^2 - 10{,}000}{x^2} \\ F'(x) &= \tfrac{(x-100)(x+100)}{x^2} \end{aligned}$$

   Hence we have critical numbers at ± 100 and $F'(x)$ is undefined at $x = 0$ (which is not in the domain of $F(x)$, so $x = 0$ is not a critical number). Normally all three numbers would go on our number line, but since we need $x > 0$, we only have to check those values. Notice our number line shows we have a local minimum at $x = 100$ and for $x > 0$, this is an absolute minimum.

   Hence the minimum amount of fencing needed is $F(100) = 100 + \frac{10{,}000}{100} = 200$ft.

4. Suppose $f(x) = \dfrac{x}{x^2 - x}$.

Domain $= \{x \mid x \neq 0, 1\}$

There is no y-intercept since $x = 0$ is not in the domain.

Setting $y = 0$, we find that there is no x-intercept either since the only value of x that makes the numerator equal to 0 is $x = 0$ which again is not in the domain.

The function is not even since $f(-x) \neq \frac{-x}{x^2+x} \neq f(x)$ and the function is not odd since $-f(-x) \neq \frac{x}{x^2+x} \neq f(x)$.

The function has a vertical asymptote at $x = 1$ since
$\displaystyle\lim_{x \to 1^-} \frac{x}{x^2 - x} = -\infty$ (and
$\displaystyle\lim_{x \to 1^+} \frac{x}{x^2 - x} = \infty$)

The function does NOT have a vertical asymptote at $x = 0$ since
$\displaystyle\lim_{x \to 0} \frac{x}{x^2 - x} = \lim_{x \to 0} \frac{1}{x - 1} = -1$
(Note there is a hole at $(0, -1)$.)

The function has a horizontal asymptote at $y = 0$ since
$\displaystyle\lim_{x \to -\infty} \frac{x}{x^2 - x} = 0$ and
$\displaystyle\lim_{x \to \infty} \frac{x}{x^2 - x} = 0$

For the remainder of the problem, we will use the fact that $f(x) = \frac{x}{x^2-x} = \frac{1}{x-1} = (x-1)^{-1}$ everywhere except $x = 0$ and take our derivatives of this reduced version instead.

Thus $f'(x) = -(x-1)^{-2}(1) = \frac{-1}{(x-1)^2}$ when $x \neq 0$. Notice there are no critical points because $f'(x) \neq 0$ and is only undefined where $f(x)$ is also undefined.

So $f(x)$ is decreasing on $(-\infty, 0) \cup (0, 1) \cup (1, \infty)$. There are no local extrema.

Notice $f''(x) = 2 \cdot (x-1)^{-3}(1) = \frac{2}{(x-1)^3}$ when $x \neq 0$. Notice there are no possible inflection points because $f''(x) \neq 0$ and is only undefined where $f(x)$ is also undefined.

So $f(x)$ is concave down on $(-\infty, 0) \cup (0, 1)$ and
$f(x)$ is concave up on $(1, \infty)$. There are no inflection points.

5. Find the most general antiderivative of

   (a) $f(\theta) = \sec^2\theta - 5\mathrm{sech}\theta\tanh\theta - \pi$

   $F(\theta) = \tan\theta + 5\mathrm{sech}\theta - \pi\cdot\theta + c$

   (b) $f(x) = \dfrac{1}{\sqrt[3]{4x}} + \dfrac{1}{1+x^2} + \dfrac{1}{x}$

   $f(x) = \dfrac{1}{\sqrt[3]{4}}x^{\frac{-1}{3}} + \dfrac{1}{1+x^2} + \dfrac{1}{x}$

   $f(x) = \dfrac{3x^{\frac{2}{3}}}{2\sqrt[3]{4}} + \tan^{-1}x + \ln|x| + c$

6. (10 pts) Verify that the function $f(x) = x^3 - 4$ satisfies the conditions of the Mean Value theorem on the interval $[-3, 0]$. Then find all the numbers c that satisfy the conclusion.

   First notice that $f(x) = x^3 - 4$ is continuous on $[-3, 0]$ since it is a polynomial.
   Also $f'(x) = 3x^2$ which exists everywhere, so $f(x)$ is differentiable on $(-3, 0)$.
   Hence $f(x) = x^3 - 4$ satisfies the hypotheses of the Mean Value Theorem on the interval $[-3, 0]$.

   The Mean Value Theorem guarantees at least one value $c \in (-3, 0)$ such that $f'(c) = \frac{f(0)-f(-3)}{0-(-3)}$.

   $$\begin{aligned} f'(c) &= \tfrac{f(0)-f(-3)}{0-(-3)} \\ 3c^2 &= \tfrac{-4-(-31)}{3} \\ 3c^2 &= \tfrac{27}{3} \\ 3c^2 &= 9 \\ c^2 &= 3 \\ c &= \pm\sqrt{3} \end{aligned}$$

   However, only $c = -\sqrt{3}$ is in the interval $(-3, 0)$, so it is the only value we were guaranteed by the Mean Value Theorem.

7. (10 pts) Estimate the area under $f(x) = x^2 - 4$ from $x = 2$ to $x = 8$ using three approximating rectangles and left endpoints. Draw the rectangles used to find this approximation.

   First notice $\Delta x = \frac{8-2}{3} = 2$.

   $$\begin{aligned} L_3 &= f(2)\cdot\Delta x + f(4)\cdot\Delta x + f(6)\cdot\Delta x \\ L_3 &= (2^2-4)\cdot\Delta 2 + (4^2-4)\cdot 2 + (6^2-4)\cdot 2 \\ L_3 &= 0\cdot\Delta 2 + 12\cdot 2 + 32\cdot 2 \\ L_3 &= 0 + 24 + 64 \\ L_3 &= 88 \end{aligned}$$