Exam 2 MTH 201 Fall 2013 Solutions

1. Given $f(x) = 5\sin(8x)$, we know

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{5\sin(8(x+h)) - 5\sin(8x)}{h} \\
&= \lim_{h\to 0} \frac{5\sin(8x+8h) - 5\sin(8x)}{h} \\
&= \lim_{h\to 0} \frac{5[\sin(8x)\cos(8h) + \sin(8h)\cos(8x)] - 5\sin(8x)}{h} \\
&= \lim_{h\to 0} \frac{5\sin(8x)\cos(8h) - 5\sin(8x)}{h} + \lim_{h\to 0} \frac{5\sin(8h)\cos(8x)}{h} \\
&= \lim_{h\to 0} \frac{5\sin(8x)(\cos(8h) - 1)}{h} + 5\cos(8x) \lim_{h\to 0} \frac{\sin(8h)}{h} \\
&= 5\sin(8x) \lim_{h\to 0} \frac{(\cos(8h) - 1)}{h} + 5\cos(8x) \cdot 8 \lim_{h\to 0} \frac{\sin(8h)}{8h} \\
&= 5\sin(8x) \cdot 8 \lim_{h\to 0} \frac{(\cos(8h) - 1)}{8h} + 5\cos(8x) \cdot 8 \cdot 1 \\
&= 5\sin(8x) \cdot 8 \cdot 0 + 40\cos(8x) \\
&= 40\cos(8x)
\end{aligned}
$$

2. (a)

$$
\begin{aligned}
\frac{dg}{dt} &= 5 \cdot 9 \cdot \cos^8(2t^3 - 4t) \cdot (-\sin(2t^3 - 4t)) \cdot (6t^2 - 4) + 14 \cdot \tan^{13}(e^t) \cdot \sec^2(e^t) \cdot e^t - e^{4-6t^2} \cdot (-12t) \\
&= -45(6t^2 - 4) \cdot \cos^8(2t^3 - 4t) \cdot \sin(2t^3 - 4t) + 14e^t \cdot \tan^{13}(e^t) \cdot \sec^2(e^t) + 12te^{4-6t^2} \\
&= -90(3t^2 - 2) \cdot \cos^8(2t^3 - 4t) \cdot \sin(2t^3 - 4t) + 14e^t \cdot \tan^{13}(e^t) \cdot \sec^2(e^t) + 12te^{4-6t^2}
\end{aligned}
$$

(b)

$$
\begin{aligned}
y' &= \frac{(10x+2) \cdot \frac{1}{2}(x^2 - x)^{\frac{-1}{2}} \cdot (2x - 1) - (x^2 - x)^{\frac{1}{2}} \cdot 10}{(10x+2)^2} \\
&= \frac{(10x+2) \cdot \frac{1}{2} \cdot (2x - 1) - (x^2 - x)^{\frac{1}{2}} \cdot 10 \cdot (x^2 - x)^{\frac{1}{2}}}{(10x+2)^2 \cdot (x^2 - x)^{\frac{1}{2}}} \\
&= \frac{(5x+1) \cdot (2x - 1) - 10(x^2 - x)}{(10x+2)^2\sqrt{x^2 - x}} \\
&= \frac{(10x^2 - 3x - 1) - (10x^2 - 10x)}{(10x+2)^2\sqrt{x^2 - x}} \\
&= \frac{7x - 1}{(10x+2)^2\sqrt{x^2 - x}}
\end{aligned}
$$

3. (a) $f'(w) = 3\ln 5 \cdot w^{\ln 5 - 1} + 1 + 0 + \pi^w \ln \pi - 4e^w$

(b) $y = x \cdot 4\sin^3(2x)\cos(2x) \cdot 2 + \sin^4(2x) \cdot 1 + \frac{1}{1+(3x)^2} \cdot 3 + \frac{-1}{\sqrt{1-(4x^3+x)^2}} \cdot (12x^2+1)$

(c) Notice

$$\begin{aligned}
f(z) &= \ln\left(\frac{(3z^2-2z+1)^5\sqrt{4z+8}}{ze^{3z}\sin z}\right) \\
&= \ln[(3z^2-2z+1)^5\sqrt{4z+8}] - \ln[ze^{3z}\sin z] \\
&= \ln(3z^2-2z+1)^5 + \ln(4z+8)^{\frac{1}{2}} - \ln z - \ln e^{3z} - \ln(\sin z) \\
&= 5\ln(3z^2-2z+1) + \frac{1}{2}\ln(4z+8) - \ln z - 3z - \ln(\sin z)
\end{aligned}$$

So $f'(z) = 5 \cdot \frac{1}{3z^2-2z+1} \cdot (6z-2) + \frac{1}{2} \cdot \frac{1}{4z+8} \cdot 4 - \frac{1}{z} - 3 - \frac{1}{\sin z \cdot \cos z}$.

(d) $h'(\theta) = e^{\sec(\sqrt[7]{6\theta^2-14\theta})} \cdot \sec(\sqrt[7]{6\theta^2-14\theta})\tan(\sqrt[7]{6\theta^2-14\theta}) \cdot \frac{1}{7}(6\theta^2-14\theta)^{\frac{-6}{7}} \cdot (12\theta-14)$

4.

$$\begin{aligned}
y &= (\ln x)^{\ln x} \\
\ln y &= \ln[(\ln x)^{\ln x}] \\
\ln y &= \ln x \cdot \ln(\ln x) \\
\tfrac{1}{y}\tfrac{dy}{dx} &= \ln x \cdot \frac{1}{(\ln x)} \cdot \frac{1}{x} + \ln(\ln x) \cdot \frac{1}{x} \\
\tfrac{1}{y}\tfrac{dy}{dx} &= \frac{1}{x} + \ln(\ln x) \cdot \frac{1}{x} \\
\tfrac{1}{y}\tfrac{dy}{dx} &= \frac{1+\ln(\ln x)}{x} \\
\tfrac{dy}{dx} &= y \cdot \frac{1+\ln(\ln x)}{x} \\
\tfrac{dy}{dx} &= (\ln x)^{\ln x} \cdot \frac{1+\ln(\ln x)}{x}
\end{aligned}$$

5. Let x be the distance between the bottom of the ladder and the base of the wall.
Let y be the distance between the top of the ladder and the base of the wall.
We know $\frac{dy}{dt} = -2\text{ft/min}$.
We want to know $\frac{dx}{dt}$ when $y = 3\text{ft}$.

Notice x, y and the 5 ft ladder form a right triangle. So by the Pythagorean Theorem we have:

$x^2 + y^2 = 5^2$

When $y = 3\text{ft}$, we see that $x^2 + 3^2 = 5^2$ and so $x = 4\text{ft}$.

Now to find $\frac{dx}{dt}$, we use implicit differentiation.

$$\begin{aligned}
x^2 + y^2 &= 5^2 \\
2x\tfrac{dx}{dt} + 2y\tfrac{dy}{dt} &= 0 \\
2 \cdot 4 \cdot \tfrac{dx}{dt} + 2 \cdot 3 \cdot (-2) &= 0 \\
8\tfrac{dx}{dt} - 12 &= 0 \\
\tfrac{dx}{dt} &= \tfrac{12}{8} = \tfrac{3}{2}
\end{aligned}$$

Hence the the distance between the bottom of the ladder and the base of the wall is increasing at a rate of $\frac{3}{2}$ft/min.

6. (a) (10 pts) Given $e^{xy} + 7y^3 = \tan(x - y) + 1$, find $\frac{dy}{dx}$ by implicit differentiation.

$$\begin{aligned}
e^{xy}[x\frac{dy}{dx} + y \cdot 1] + 21y^2\frac{dy}{dx} &= \sec^2(x - y)[1 - \frac{dy}{dx}] \\
xe^{xy}\frac{dy}{dx} + ye^{xy} + 21y^2\frac{dy}{dx} &= \sec^2(x - y) - \sec^2(x - y)\frac{dy}{dx} \\
xe^{xy}\frac{dy}{dx} + 21y^2\frac{dy}{dx} + \sec^2(x - y)\frac{dy}{dx} &= \sec^2(x - y) - ye^{xy} \\
[xe^{xy} + 21y^2 + \sec^2(x - y)]\frac{dy}{dx} &= \sec^2(x - y) - ye^{xy} \\
\frac{dy}{dx} &= \frac{\sec^2(x - y) - ye^{xy}}{xe^{xy} + 21y^2 + \sec^2(x - y)}
\end{aligned}$$

(b) (10 pts) : Find the tangent line of equation given above at the point $(0, 0)$.

We find the slope at $(0, 0)$ by plugging in $x = 0, y = 0$ into our equation for $\frac{dy}{dx}$ and get

$$\begin{aligned}
\frac{dy}{dx} &= \frac{\sec^2(0 - 0) - 0 \cdot e^{0 \cdot 0}}{0 \cdot e^{0 \cdot 0} + 21 \cdot 0^2 + \sec^2(0 - 0)} \\
\frac{dy}{dx} &= \frac{\sec^2(0) - 0}{0 + 0 + \sec^2(0)} \\
\frac{dy}{dx} &= \frac{1^2}{1^2} \\
\frac{dy}{dx} &= 1 = m
\end{aligned}$$

Hence the tangent line is $y - 0 = 1(x - 0)$. That is, the tangent line $y = x$.