EXAM 1

Score: ________ out of 100

Math 201 - Calculus I

Name: key

Read all of the following information before starting the exam:

- You have 60 minutes to complete the exam.
- Show all work, clearly and in order, if you want to get full credit. Please make sure you read the directions for each problem. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Please $\boxed{\text{box/circle}}$ or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements.
- This test has 6 problems and is worth 100 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

1. Calculate the following limits. If the limit is ∞ or $-\infty$ clearly indicate this. Otherwise, for limits that do not exist, write D.N.E.

(a) $\lim_{x\to 4} \frac{x^2-2x-8}{x^2-16}$

$= \lim_{x\to 4} \frac{\cancel{(x-4)}(x+2)}{\cancel{(x-4)}(x+4)}$

$= \lim_{x\to 4} \frac{x+2}{x+4}$

$= \frac{4+2}{4+4} = \frac{6}{8} = \frac{3}{4}$

answer: $\boxed{3/4}$

(b) $\lim_{x\to 0} \frac{\sqrt{1+x}-1}{x}$

$= \lim_{x\to 0} \frac{(\sqrt{1+x}-1)}{x} \frac{(\sqrt{1+x}+1)}{(\sqrt{1+x}+1)}$

$= \lim_{x\to 0} \frac{(\sqrt{1+x})^2-1}{x(\sqrt{1+x}+1)}$

$= \lim_{x\to 0} \frac{1+x-1}{x(\sqrt{1+x}+1)}$

$= \lim_{x\to 0} \frac{\cancel{x}}{\cancel{x}(\sqrt{1+x}+1)} = \lim_{x\to 0} \frac{1}{\sqrt{1+x}+1} = \frac{1}{\sqrt{1}+1} = \frac{1}{1+1} = \frac{1}{2}$

answer: $\boxed{1/2}$

(c) $\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{x^2+x}\right)$

$= \lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{x(x+1)}\right)$

$= \lim_{x\to 0} \left(\frac{x+1}{x(x+1)} - \frac{1}{x(x+1)}\right)$

$= \lim_{x\to 0} \left(\frac{x+1-1}{x(x+1)}\right)$

$= \lim_{x\to 0} \left(\frac{x}{x(x+1)}\right) = \lim_{x\to 0} \left(\frac{1}{x+1}\right) = \frac{1}{0+1} = \frac{1}{1} = 1$

answer: $\boxed{1}$

(d) $\lim_{\theta \to \frac{\pi}{6}} \frac{2\sin(\theta) + \tan(6\theta)}{\cos(12\theta)}$

$$= \frac{2\sin(\pi/6) + \tan(6 \cdot \pi/6)}{\cos(12 \cdot \pi/6)}$$

$$= \frac{2(\frac{1}{2}) + \tan(\pi)}{\cos(2\pi)}$$

$$= \frac{1+0}{1} = 1$$

answer: $\boxed{1}$

(e) $\lim_{x \to 2} e^{(x^2+2x-8)}$

$$= e^{4+4-8}$$

$$= e^{0}$$

$$= 1$$

answer: $\boxed{1}$

(f) $\lim_{t \to -3^-} \frac{t+3}{|t+3|}$

$$|t+3| = \begin{cases} t+3 & \text{if } t+3 \geq 0 \\ -(t+3) & \text{if } t+3 < 0 \end{cases} = \begin{cases} t+3 & \text{if } t \geq -3 \\ -(t+3) & \text{if } t < -3 \end{cases}$$

$$= \lim_{t \to -3^-} \frac{t+3}{-(t+3)} \quad (\text{since } t < -3)$$

$$= \lim_{t \to -3^-} \frac{1}{-1}$$

$$= -1$$

answer: $\boxed{-1}$

2. Let $f(x) = \dfrac{x^2 - 1}{x^2 + 3x + 2} = \dfrac{(x-1)(x+1)}{(x+1)(x+2)}$

(a) Find the vertical asymptote(s) of f. Justify completely.

potential vertical asymptote(s) when denominator is 0:

$$(x+1)(x+2) = 0$$

$$x = -1 \quad \text{or} \quad x = -2$$

Check:

at $x=-1$ there is a hole in the graph but NO asymptote since

$$\lim_{x \to -1} f(x) = \lim_{x \to -1} \frac{(x-1)\cancel{(x+1)}}{\cancel{(x+1)}(x+2)} = \frac{-1-1}{-1+2} = \frac{-2}{1} = -2$$

at $x=-2$ we need to see if the limit from the left or the right approaches $+\infty$ or $-\infty$:

$$\lim_{x \to -2^+} f(x) = \lim_{x \to -2^+} \frac{(x-1)\cancel{(x+1)}}{\cancel{(x+1)}(x+2)} = \lim_{x \to -2^+} \frac{x-1}{x+2} = -\infty$$

goes to 3

goes to 0 and is +

This is enough information to say there is a vertical asymptote @ $x = -2$. You could also check:

$$\lim_{x \to -2^-} f(x) = +\infty$$

but this is not necessary.

answer: $\boxed{x = -2}$

(only one vertical asymptote)

(b) Find the horizontal asymptote(s) of f. Justify completely.

$$\lim_{x \to \infty} f(x) = \lim_{x \to \infty} \left(\frac{x^2 - 1}{x^2 + 3x + 2}\right) \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)}$$

$$= \lim_{x \to \infty} \frac{1 - \frac{1}{x^2}}{1 + \frac{3}{x} + \frac{2}{x^2}} = \frac{1 - 0}{1 + 0 + 0} = 1$$

so there is a horizontal asymptote at $y = 1$.

Same as $x \to -\infty$:

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \left(\frac{x^2 - 1}{x^2 + 3x + 2}\right) \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)}$$

$$= \lim_{x \to -\infty} \frac{1 - 1/x^2}{1 + \frac{3}{x} + \frac{2}{x^2}} = \frac{1 - 0}{1 + 0 + 0} = 1$$

same as before.

answer: $\boxed{y = 1}$

(only one horizontal asymptote)

3. Use the Squeeze Theorem to show that $\lim_{x\to 0} 2x^6 \cos\left(\frac{5}{x^{10}}\right) = 0$.

For any θ: $\quad -1 \le \cos\theta \le 1$,

so: $\quad -1 \le \cos\left(\frac{5}{x^{10}}\right) \le 1 \quad$ as long as $x \ne 0$.

Therefore, $\quad -2x^6 \le 2x^6 \cos\left(\frac{5}{x^{10}}\right) \le 2x^6$.

Now let's see what the left bound and right bound approach as $x \to 0$:

$$\lim_{x\to 0} (-2x^6) = -2(0)^6 = 0.$$

$$\lim_{x\to 0} (2x^6) = 2(0)^6 = 0.$$

Hence, by squeeze theorem.

$$\lim_{x\to 0} 2x^6 \cos\left(\frac{5}{x^{10}}\right) = 0.$$

4. Use the ϵ, δ definition of the limit to show that $\lim_{x\to\frac{1}{2}} (2x+1) = 2$.

Here, $a = \frac{1}{2}$, $f(x) = 2x+1$, $L = 2$.

(1) Let $\varepsilon > 0$

(2) Choose $\delta = \underline{\ \varepsilon/2\ }$

(3) Suppose x is such that $0 < |x - \frac{1}{2}| < \delta$

(4) $|f(x) - L| = |(2x+1) - 2|$

$$= |2x - 1|$$

$$= |2(x - \tfrac{1}{2})|$$

$$= 2|x - \tfrac{1}{2}| < 2\delta = 2\left(\tfrac{\varepsilon}{2}\right) = \varepsilon \qquad \square$$

(Now fill in (2))

5. Where is the function $f(x) = \dfrac{\tan^{-1}(x^2+1) - \ln(x-5)}{e^{(\sin(x)+\cos(x))}}$ continuous?.

Numerator Restrictions:

- $\tan^{-1}(\underbrace{x^2+1}_{\text{polynomial is continuous everywhere.}})$ inverse tan is continuous everywhere.
- $\ln(x-5)$ must have $x-5>0 \rightarrow \underline{x>5}$

Denominator Restrictions: sine and cosine are continuous everywhere

- $e^{\sin(x)+\cos(x)}$ e^x is continuous everywhere
- denominator $\neq 0$.
  It is NOT possible for $e^{\sin(x)+\cos(x)} = 0$ since e^x is always positive.

All restrictions together: $x>5$ (ONLY $\ln(x-5)$ introduces an issue)

answer: $\boxed{x>5}$

6. Use the Intermediate Value Theorem to show that the equation $\cos(5x) = 8x^3$ has a root in the interval $(0, \frac{1}{2})$.

Consider $\underbrace{\cos(5x) - 8x^3}_{f(x)} = 0$

Now,

$$f(0) = \cos(5\cdot 0) - 8\cdot 0^3 = \cos(0) - 0 = 1 > 0$$

$$f(\tfrac{1}{2}) = \cos(\tfrac{5}{2}) - 8(\tfrac{1}{2})^3 = \underbrace{\cos(\tfrac{5}{2})}_{\text{less than 1}} - 1 < 0$$

Therefore, since f is continuous on $(0, 1/2)$ (this is because cosine and polynomials are cts. everywhere)

the Intermediate Value Theorem guarantees there is an x in $(0,\frac{1}{2})$ such that $f(x) = 0$.