EXAM 3

Score: ________ out of 100

Math 201 - Calculus I

Name: Key

Read all of the following information before starting the exam:

- You have 60 minutes to complete the exam.
- Show all work, clearly and in order, if you want to get full credit. Please make sure you read the directions for each problem. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Please $\boxed{\text{box/circle}}$ or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements.
- This test has 5 problems and is worth 100 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

1. Find all numbers c that satisfy the conclusion of the Mean Value Theorem for

$$f(x) = 3x^2 + 2x + 5,$$

on the interval $[-1, 1]$.

$$f'(x) = 6x + 2$$

Mean Value Thm tells us there exists a c in $[-1,1]$ such that

$$f'(c) = \frac{f(b)-f(a)}{b-a} = \frac{f(1)-f(-1)}{1-(-1)}$$

$$6c + 2 = \frac{(3+2+5)-(3-2+5)}{2}$$

$$6c + 2 = \frac{10-6}{2} = \frac{4}{2} = 2$$

$$6c + 2 = 2$$

$$6c = 0$$

$$c = 0$$

c **value(s):** $\boxed{c = 0}$

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2. Find the most general antiderivative of the following functions:

(a) $f(x) = x^3 + \dfrac{1}{x} - \dfrac{2}{1+x^2} + \cos(x) + 3\sec^2(x)$

answer: $\boxed{F(x) = \frac{x^4}{4} + \ln|x| - 2\tan^{-1}(x) + \sin(x) + 3\tan(x) + C}$

(b) $f(x) = \dfrac{1}{\sqrt{x}} - \dfrac{1}{x^3} + \dfrac{x^3 - x}{x}$

$$= x^{-1/2} - x^{-3} + \frac{x^3}{x} - \frac{x}{x}$$

$$= x^{-1/2} - x^{-3} + x^2 - 1 \quad \text{so}$$

answer: $\boxed{F(x) = \frac{x^{1/2}}{1/2} - \frac{x^{-2}}{-2} + \frac{x^3}{3} - x + C}$ OR

$\boxed{F(x) = 2\sqrt{x} + \frac{1}{2x^2} + \frac{x^3}{3} - x + C}$

3. Calculate the following limits. If the limit is ∞ or $-\infty$ clearly indicate this. Otherwise, for limits that do not exist, write D.N.E.

(a) $\lim_{x\to 0} \frac{\sin(4x)}{\tan(5x)}$ ← Type: $\frac{0}{0}$ (indeterminant form.)

$$\overset{L'H}{=} \lim_{x\to 0} \frac{\frac{d}{dx}\sin(4x)}{\frac{d}{dx}\tan(5x)}$$

$$= \lim_{x\to 0} \frac{4\cos(4x)}{5\sec^2(5x)}$$

$$= \frac{4\cos(0)}{5\sec^2(0)} = \frac{4\cdot 1}{5\cdot 1} = \frac{4}{5}$$

answer: $\boxed{4/5}$

SOL 2: (LONG).

$$\lim_{x\to 0} \frac{\sin(4x)}{\tan(5x)} = \lim_{x\to 0} \frac{\sin(4x)}{\left(\frac{\sin(5x)}{\cos(5x)}\right)}$$

$$= \lim_{x\to 0} \sin(4x)\left(\frac{\cos(5x)}{\sin(5x)}\right)$$

$$= \lim_{x\to 0} \frac{\sin(4x)}{\sin(5x)} \cdot 1$$

$$\overset{L'H}{=} \lim_{x\to 0} \frac{4\cos(4x)}{5\cos(5x)}$$

$$= \boxed{4/5}$$

(b) $\lim_{x\to\infty} \frac{x+x^2}{1-2x^2}$ ← Type $\frac{\infty}{-\infty}$ (indeterminant form)

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{1+2x}{0-4x}$$

$$\overset{L'H}{=} \lim_{x\to\infty} \frac{2}{-4}$$

$$= -1/2$$

answer: $\boxed{-1/2}$

SOL 2:

$$\lim_{x\to\infty} \left(\frac{x+x^2}{1-2x^2}\right) \frac{\left(\frac{1}{x^2}\right)}{\left(\frac{1}{x^2}\right)}$$

$$= \lim_{x\to\infty} \frac{\frac{1}{x}+1}{\frac{1}{x^2}-2}$$

$$= \frac{0+1}{0-2} = \boxed{-\frac{1}{2}}$$

(c) $\lim_{x\to 0^+} x^x$

$$= \lim_{x\to 0^+} e^{\ln(x^x)}$$

$$= \lim_{x\to 0^+} e^{x\ln(x)} = e^{\lim_{x\to 0^+} x\ln(x)} = e^{\lim_{x\to 0^+} \frac{\ln x}{\left(\frac{1}{x}\right)}}$$

goes to 0

goes to $-\infty$.

goes to $-\infty$

goes to ∞

$$\overset{L'H}{=} e^{\lim_{x\to 0^+} \frac{\left(\frac{1}{x}\right)}{\left(-\frac{1}{x^2}\right)}} = e^{\lim_{x\to 0^+}(-x)} = e^0 = 1$$

answer: $\boxed{1}$

4. Let $f(x) = \frac{1}{x^2-9}$. **If you find that f does not have something that is asked for, then please clearly write NONE (or some other indication of this).**

(a) **Domain of** f: $\boxed{x \neq 3, x \neq -3 \text{ i.e., } (-\infty,-3)\cup(-3,3)\cup(3,\infty)}$

(b) y**-intercept (if any):** $\boxed{(0, f(0)) = (0, -1/9)}$

(c) x**-intercept(s) (if any):** $\boxed{\text{NONE}}$ $\frac{1}{x^2-9} = 0$ has NO Solution since $1 \neq 0$.

(d) Analyze the **symmetry** of f by determining whether f is **even, odd** or **neither**. Justify completely.

$$f(-x) = \frac{1}{(-x)^2-9} = \frac{1}{x^2-9} = f(x)$$

Hence, f is even

f is: $\boxed{\text{EVEN}}$

(e) Find the **horizontal asymptote(s)** of f. Justify completely.

$$\lim_{x\to\infty} \frac{1}{x^2-9} = 0$$

← numerator goes to 1

← denom. goes to ∞

$$\lim_{x\to-\infty} \frac{1}{x^2-9} = 0$$

horizontal asymptote(s) (if any): $\boxed{y = 0}$

(f) Find the **vertical asymptote(s)** of f. Justify completely and determine the behavior on both sides of each vertical asymptote.

$$f(x) = \frac{1}{x^2-9} = \frac{1}{(x-3)(x+3)}$$

The denominator is 0 if $x=3$, or $x=-3$ so These are potential vertical asymptotes.

$$\lim_{x\to3^+} \frac{1}{(x-3)(x+3)} \overset{(+)}{=} \frac{(+)}{(+)(+)} \quad \boxed{+\infty}$$

$$\lim_{x\to-3^+} \frac{1}{(x-3)(x+3)} \overset{(+)}{=} \frac{(+)}{(-)(+)} \quad \boxed{-\infty}$$

$$\lim_{x\to3^-} \frac{1}{(x-3)(x+3)} = \frac{(+)}{(-)(+)} \quad \boxed{-\infty}$$

$$\lim_{x\to-3^-} \frac{1}{(x-3)(x+3)} = \frac{(+)}{(-)(-)} \quad \boxed{+\infty}$$

vertical asymptote(s) (if any): $\boxed{x=3, \ x=-3}$

(g) Determine where f is **increasing**, **decreasing** and indicate the **local maximum value(s)** and **local minimum value(s)**.

$$f'(x) = \frac{(x^2-9)0 - 1(2x)}{(x^2-9)^2} = \frac{-2x}{(x^2-9)^2}$$

$f'(x) = 0$
$-2x = 0$
$\underline{x = 0}$.

$f'(x)$ does not exis if $x = 3$ or -3 BUT these valus are NOT in the doman of f.

f': $\frac{+}{+} = +$ on $(-\infty,-3)$; -3 (open); $\frac{+}{+} = +$ on $(-3,0)$; 0 (closed); $\frac{-}{+} = -$ on $(0,3)$; 3 (open); $\frac{-}{+} = -$ on $(3,\infty)$ ← domain of f

intervals: $(-\infty,-3)$ -3 $(-3,0)$ 0 $(0,3)$ 3 $(3,\infty)$

Intervals where f is increasing (if any): $(-\infty,-3)$ AND $(-3,0)$

Intervals where f is decreasing (if any): $(0,3)$ AND $(3,\infty)$

Local maximum value(s) (if any): $(0, -1/9)$

Local minimum values(s) (if any): NONE

(h) Determine where f is **concave up**, **concave down** and indicate the **inflection point(s)**.

$$f''(x) = \frac{(x^2-9)^2(-2) - (-2x)\left[2(x^2-9)'(2x)\right]}{(x^2-9)^4}$$

$$= \frac{-2(x^2-9)\left[(x^2-9) - 2x(2x)\right]}{(x^2-9)^4} = \frac{-2\left[x^2-9-4x^2\right]}{(x^2-9)^3} = \frac{-2(-3x^2-9)}{(x^2-9)^3}$$

$$= \frac{2(3x^2+9)}{(x^2-9)^3}$$

NEVER is this 0 (if you try to solve $f''(x)=0$ you get complex roots.) since $3x^2+9=0 \rightarrow x = \pm\sqrt{-3}$

f'': $\frac{+}{+} = +$ on $(-\infty,-3)$; -3 (open); $\frac{+}{-} = -$ on $(-3,3)$; 3 (open); $+$ on $(3,\infty)$ ← domain of f.

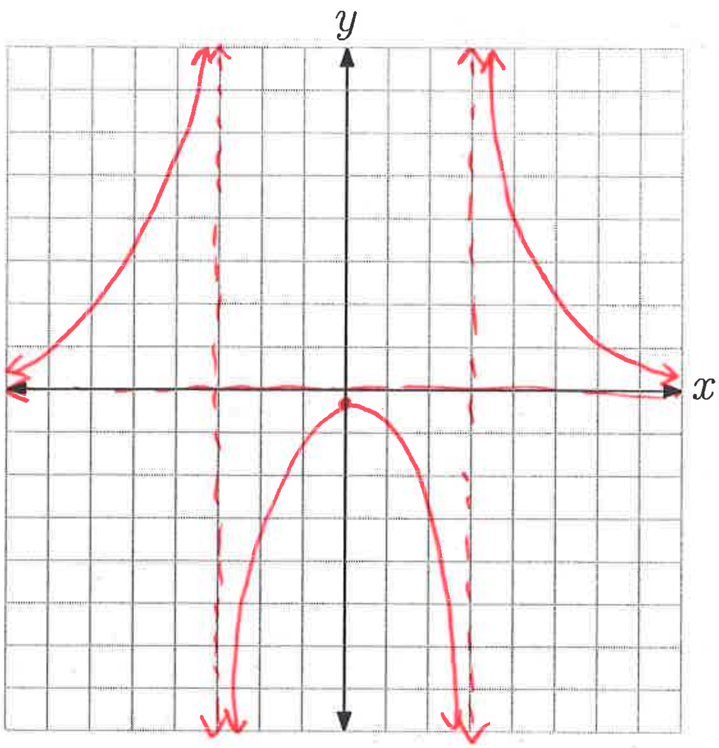
intervals: $(-\infty,-3)$ -3 $(-3,3)$ 3 $(3,\infty)$

Intervals where f is concave up (if any): $(-\infty,-3)$ AND $(3,\infty)$

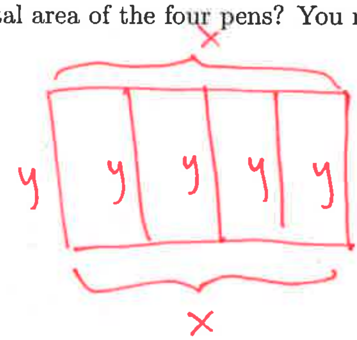
Intervals where f is concave down (if any): $(-3,3)$

Inflection point(s) (if any): NONE

(i) Sketch the graph of $y = f(x)$



5. A farmer with 500 ft of fencing wants to enclose a rectangular area and then divide it into four pens with fencing parallel to one side of the rectangle. What are the dimensions of the largest possible total area of the four pens? You must fully justify your answer using calculus.



$500\text{ft} = 2x + 5y \longrightarrow y = \dfrac{500 - 2x}{5}$

Total Area $= A = xy = x\left(\dfrac{500-2x}{5}\right)$

$A = x\left(100 - \frac{2}{5}x\right)$

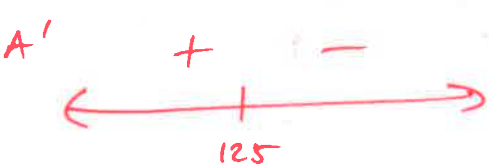
$A = 100x - \frac{2}{5}x^2$

$A'(x) = 100 - 4/5\,x$

$A'(x) = 0 = 100 - 4/5\,x$

$100 = 4/5\,x$

$\frac{500}{4} = x = 125$



abs max if $x = 125$

$y = 100 - \frac{2}{5}(125)$

$= 100 - 50$

$= 50$

Dimensions (or give the total area): 125ft × 50ft