① If $\vec{y} \in \text{span}(\{\vec{x}_1, \dots, \vec{x}_n)$, then $\exists c_1, c_2, \dots, c_n \in \mathbb{R}$ such that

$$\vec{y} = c_1\vec{x}_1 + c_2\vec{x}_2 + \cdots + c_n\vec{x}_n .$$

(i.e., $\vec{y}$ can be written as a linear combination of the $\vec{x}_i$'s)

Subtract everything to the left :

$$\vec{y} - c_1\vec{x}_1 - c_2\vec{x}_2 - \cdots - c_n\vec{x}_n = \vec{0} .$$

This is a nontrivial linear combination of $\vec{y}, \vec{x}_1, \dots, \vec{x}_n$ since the coefficient of $\vec{y}$ is 1. Hence, by definition $\{\vec{y}, \vec{x}_1, \dots, \vec{x}_n\}$ is linearly dependent. $\square$

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② $W = \left\{ \begin{bmatrix} x_1 \\ 0 \\ 0 \\ x_4 \end{bmatrix} \middle| \; x_1, x_4 \in \mathbb{R} \right\}$

Is W a subspace of $\mathbb{R}^4$?

(i) $\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \in W$ ✓

(ii) Let $\begin{bmatrix} x_1 \\ 0 \\ 0 \\ x_4 \end{bmatrix}, \begin{bmatrix} y_1 \\ 0 \\ 0 \\ y_4 \end{bmatrix} \in W$.

$$\begin{bmatrix} x_1 \\ 0 \\ 0 \\ x_4 \end{bmatrix} + \begin{bmatrix} y_1 \\ 0 \\ 0 \\ y_4 \end{bmatrix} = \begin{bmatrix} x_1+y_1 \\ 0 \\ 0 \\ x_4+y_4 \end{bmatrix} \in W \quad ✓$$

(iii) Let $\begin{bmatrix} x_1 \\ 0 \\ 0 \\ x_4 \end{bmatrix} \in W$, $k \in \mathbb{R}$

$$k\begin{bmatrix} x_1 \\ 0 \\ 0 \\ x_4 \end{bmatrix} = \begin{bmatrix} kx_1 \\ 0 \\ 0 \\ kx_4 \end{bmatrix} \in W \quad ✓$$

Yes, W is a subspace of $\mathbb{R}^4$.

(3) Is $W = \{a_0 + a_3x^3 \mid a_0, a_3 \in \mathbb{R}\}$ a subspace of P_3?

(i) The $\vec{0}$ in P_3 is the polynomial: 0
since we can write $0 = 0 + 0x^3$, 0 is in W. ✓

(ii) Suppose $a_0 + a_3x^3 \in W$
$b_0 + b_3x^3 \in W$

$$(a_0 + a_3x^3) + (b_0 + b_3x^3) = (a_0 + b_0) + (a_3 + b_3)x^3 \in W$$ ✓

(iii) Suppose $a_0 + a_3x^3 \in W$, and $k \in \mathbb{R}$

$$k(a_0 + a_3x^3) = (ka_0) + (ka_3)x^3 \in W$$ ✓

Yes, W is a subspace of P_3.

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(4) see next page.

(5) Is $W = \{A \in M_{33} \mid tr(A) = 4\}$ a subspace of M_{33}?

SOL:
NO!

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \in W \quad \text{but}$$

$$\begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 4 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 8 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

trace $= 8 + 0 + 0 = 8 \neq 4$.

So $\begin{bmatrix} 8 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \notin W$.

Not closed under vector addition.

(4) Is $W = \{A \in M_{33} \mid tr(A) = 0\}$ a subspace of M_{33}?

(i) $\vec{0}$ in M_{33} is $\begin{bmatrix} 0&0&0\\0&0&0\\0&0&0 \end{bmatrix}$ and since this has trace $= 0$ it is in W ✓

(ii) Let $A, B \in W$.

$A + B \in M_{33}$ but if $tr(A) = 0$ AND $tr(B) = 0$
we have $tr(A+B) = tr(A) + tr(B) = 0 + 0 = 0$ ✓
Hence, $A + B \in W$.

(iii) Let $A \in W$, $k \in \mathbb{R}$
so $tr(A) = 0$
$tr(kA) = k\,tr(A) = k \cdot 0 = 0$ so $kA \in W$. ✓

Yes! W is a subspace of M_{33}.

(5) see previous page.

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(6) Is $W = \{f(x) = A\sin(x) + B\cos(x) \mid A, B \in \mathbb{R}\}$ a subspace of $F(-\infty, \infty)$?

(i) The zero function $f(x) = 0$ can be expressed as
$f(x) = 0 = 0\sin(x) + 0\cos(x)$
hence $0 \in W$ ✓

Let $f(x) = A\sin(x) + B\cos(x) \in W$
$g(x) = C\sin(x) + D\cos(x) \in W$
$k \in \mathbb{R}$

(ii) $f(x) + g(x) = \underbrace{(A+C)}_{\in \mathbb{R}}\sin(x) + \underbrace{(B+D)}_{\in \mathbb{R}}\cos(x) \in W$ ✓

(iii) $kf(x) = \underbrace{(kA)}_{\in \mathbb{R}}\sin(x) + \underbrace{(kB)}_{\in \mathbb{R}}\cos(x) \in W$ ✓

YES! W is a subspace of $F(-\infty, \infty)$

Anyone that has taken Diffeq might recognize what W is (solution space to some linear ODE) do you see which one?

7. Let U and W be subspaces of a vector space V. Is $U \cap W = \{x \mid x \in U \text{ and } x \in W\}$ a subspace of V?

(i) Since U and W are subspaces of V, both contain $\vec{0} \in V$. i.e., $\vec{0} \in U$ and $\vec{0} \in W$.
Hence $\vec{0} \in U \cap W$. ✓

(ii) Let $\vec{x}, \vec{y} \in U \cap W$.

So $\vec{x} \in U$ and $\vec{x} \in W$.
$\vec{y} \in U$ and $\vec{y} \in W$.

Now, since U and W are subspaces we know

$$\vec{x} + \vec{y} \in U \quad \text{and} \quad \vec{x} + \vec{y} \in W.$$

Hence, $\vec{x}+\vec{y} \in U \cap W$. ✓

(iii) Let $\vec{x} \in U \cap W$
Let $\vec{k} \in \mathbb{R}$.

So $\vec{x} \in U$ and $\vec{x} \in W$.

Now, since U and W are subspaces we know

$$k\vec{x} \in U \quad \text{and} \quad k\vec{x} \in W.$$

Hence, $k\vec{x} \in U \cap W$. ✓

Yes! $U \cap W$ is a subspace of V □

7 continued...

Is $\mathcal{U} \cup W = \{x \mid x \in \mathcal{U} \text{ OR } x \in W\}$ a subspace of V?

$\boxed{\text{SOL 1:}}$ (specific example)

~~Not always~~. Consider $V = \mathbb{R}^2$.

consider $\mathcal{U} = \text{span}\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\} = \left\{\begin{bmatrix}0\\x_2\end{bmatrix} \,\middle|\, x_2 \in \mathbb{R}\right\}$

$W = \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\} = \left\{\begin{bmatrix}x_1\\0\end{bmatrix} \,\middle|\, x_1 \in \mathbb{R}\right\}$

now $\begin{bmatrix}0\\1\end{bmatrix} \in \mathcal{U} \cup W$ (since $\begin{bmatrix}0\\1\end{bmatrix} \in \mathcal{U}$)

$\begin{bmatrix}1\\0\end{bmatrix} \in \mathcal{U} \cup W$ (since $\begin{bmatrix}1\\0\end{bmatrix} \in W$)

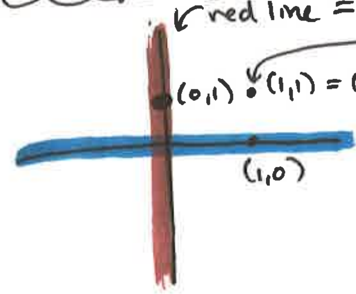
but $\begin{bmatrix}0\\1\end{bmatrix} + \begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\1\end{bmatrix} \notin \mathcal{U} \cup W$ because $\begin{bmatrix}1\\1\end{bmatrix} \notin \mathcal{U}$ and $\begin{bmatrix}1\\1\end{bmatrix} \notin W$.

hence this bit shows $\mathcal{U} \cup W$ is not closed under vector addition.

answer: $\boxed{\text{NO}}$ □

Some geometry:

red line = $\mathcal{U} = \text{span}\left\{\begin{bmatrix}0\\1\end{bmatrix}\right\}$



this is not on the red line OR the blue line.

blue line = $W = \text{span}\left\{\begin{bmatrix}1\\0\end{bmatrix}\right\}$

$\mathcal{U} \cup W$ = points on either the red line OR the blue line.

7 continued ....

Is $U+W = \{\vec{x}+\vec{y} \mid \vec{x} \in U \text{ and } \vec{y} \in W\}$ a subspace of V?

(8) Let V be a vector space.

Suppose $S, T \subseteq V$. (not necessarily subspaces.)

(a) Show:

If $S \subseteq T$, then $\text{span}(S) \subseteq \text{span}(T)$.

proof: For simplicity lets say $S = \{x_1, \dots, x_n\}$ and $T = \{x_1, \dots, x_n, x_{n+1}, \dots, x_p\}$

Suppose $S \subseteq T$.

(Show: $\text{span}(S) \subseteq \text{span}(T)$)

Let $\vec{x} \in \text{span}(S) \longrightarrow \exists \alpha_1, \dots, \alpha_n \in \mathbb{R}$ s.t.

$$x = \alpha_1 \vec{x}_1 + \dots + \alpha_n \vec{x}_n$$

so $$x = \alpha_1 \vec{x}_1 + \dots + \alpha_n \vec{x}_n + 0\vec{x}_{n+1} + \dots + 0\vec{x}_p$$

Hence, $\vec{x} \in \text{span}(T)$. $\square$.

(b). $\text{span}(\text{span}(S)) = \text{span}(S)$.

proof. ~~Lemma:~~

mini lemma.

Notice that for any set $S = \{\vec{x}_1, \dots, \vec{x}_n\}$

$$S \subseteq \text{span}(S)$$

~~proof of lemma:~~

Since each x_i can be expressed as

$$x_i = 0\vec{x}_1 + \dots + 0\vec{x}_{i-1} + \vec{x}_i + 0\vec{x}_{i+1} + \dots + 0\vec{x}_n$$

so $x_i \in \text{span}(S)$.

To prove (b) we can (show ($\subseteq$) and ($\supseteq$) to get equality)

i.e. Show (i) $\text{span}(\text{span}(S)) \subseteq \text{span}(S)$ AND

(ii) $\text{span}(S) \subseteq \text{span}(\text{span}(S))$ then the result follows $\rightarrow$

(i) $\text{span}(\text{span}(S)) \subseteq \text{span}(S)$

Let $\vec{x} \in \text{span}(\text{span}(S))$

$\longrightarrow$ $\vec{x}$ can be written as a linear combination of the vectors in span(S)

but all the vectors in span(S) can be written as a lin. combination

of the vectors in S.

hence $\vec{x}$ can be written as a linear combination of the vectors in S.

so $\vec{x} \in \text{span}(S)$.

(ii) by the mini lemma

$$S \subseteq \underbrace{\text{span}(S)}_{\text{treat this like } T.}$$

now by (a)

$$\text{span}(S) \subseteq \text{span}(\text{span}(S))$$