COMMENTS FOR WEEK 1

Here are some practice examples with RREF. Please try to do them on your own and then check your answers! Also, you may choose to do some row operations in a different order, which is okay, but the goal is still to get the matrix into RREF. If you prefer to put the matrix in REF and solve by hand and back substitute, that is also okay, but you should still get the same final answers.

Example 1:

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ -1 & 2 & -1 & 1 \\ 2 & 1 & 3 & 0 \end{array}\right] \xrightarrow{R2\to R2+R1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & 3 \\ 2 & 1 & 3 & 0 \end{array}\right] \xrightarrow{R3\to R3-2R1} \left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 3 & 1 & -4 \end{array}\right] \xrightarrow{R3\to R3-3R2}$$

$$\left[\begin{array}{ccc|c} 1 & -1 & 1 & 2 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -13 \end{array}\right] \xrightarrow{R1\to R1+R2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 5 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -13 \end{array}\right] \xrightarrow{R1\to R1-R3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 18 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -13 \end{array}\right]$$

Hence, the linear system has one solution: $\boxed{x_1 = 18,\ x_2 = 3 \text{ and } x_3 = 13}$.

Example 2:

When you are doing the row operations, you are allowed to do more than one at a time as long as they are very clear.

$$\left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 3 \\ 1 & -1 & 1 & 0 & 5 \\ 2 & -2 & 1 & 0 & 8 \end{array}\right] \xrightarrow[R3\to R3-2R1]{R2\to R2-R1} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 & 2 \end{array}\right] \xrightarrow{R3\to R3-R2} \left[\begin{array}{cccc|c} 1 & -1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right]$$

Therefore, the leading variables are x_1 and x_3, while the free variables are x_2 and x_4. Translating the reduced augmented matrix:

$$\begin{aligned} x_1 &= 3 + x_2 \\ x_3 &= 2 \end{aligned}$$

We will assign parameters $x_2 = s$ and $x_4 = t$ to get the following general solution (parametric equations):

$$\boxed{\begin{aligned} x_1 &= 3 + s \\ x_2 &= s \\ x_3 &= 2 \\ x_4 &= t \end{aligned}}$$

An alternative representation for the general solution is the following:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ 2 \\ 0 \end{bmatrix} + s\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

Example 3:

$$\left[\begin{array}{ccc|c} 1 & 2 & 0 & 3 \\ -1 & -2 & 1 & 1 \\ -1 & -2 & 0 & 5 \end{array}\right] \xrightarrow{\text{Putting into RREF}} \left[\begin{array}{ccc|c} 1 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{array}\right]$$

Therefore, the system has $\boxed{\text{no solution}}$ because the last row translates to 0=1, which is impossible.

Example 4 (Good practice with fractions):

$$\left[\begin{array}{ccc|c} 3 & 1 & -1 & 10 \\ 2 & 1 & 2 & 5 \\ -2 & 2 & 3 & 1 \end{array}\right] \xrightarrow{R2\to R2-\frac{2}{3}R1} \left[\begin{array}{ccc|c} 3 & 1 & -1 & 10 \\ 0 & 1/3 & 8/3 & -5/3 \\ -2 & 2 & 3 & 1 \end{array}\right] \xrightarrow{R3\to R3+\frac{2}{3}R1}$$

$$\left[\begin{array}{ccc|c} 3 & 1 & -1 & 10 \\ 0 & 1/3 & 8/3 & -5/3 \\ 0 & 8/3 & 7/3 & \frac{23}{3} \end{array}\right] \xrightarrow{R3\to R3-8R2} \left[\begin{array}{ccc|c} 3 & 1 & -1 & 10 \\ 0 & 1/3 & 8/3 & -5/3 \\ 0 & 0 & -19 & 21 \end{array}\right] \xrightarrow{R1\to \frac{1}{3}R1}$$

$$\left[\begin{array}{ccc|c} 1 & 1/3 & -1/3 & 10/3 \\ 0 & 1/3 & 8/3 & -5/3 \\ 0 & 0 & -19 & 21 \end{array}\right] \xrightarrow{R2\to 3R2} \left[\begin{array}{ccc|c} 1 & 1/3 & -1/3 & 10/3 \\ 0 & 1 & 8 & -5 \\ 0 & 0 & -19 & 21 \end{array}\right] \xrightarrow{R1\to R1-\frac{1}{3}R2}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & -3 & 5 \\ 0 & 1 & 8 & -5 \\ 0 & 0 & -19 & 21 \end{array}\right] \xrightarrow{R3\to \frac{-1}{19}R3} \left[\begin{array}{ccc|c} 1 & 0 & -3 & 5 \\ 0 & 1 & 8 & -5 \\ 0 & 0 & 1 & -\frac{21}{19} \end{array}\right] \xrightarrow{R1\to R1+3R3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{32}{19} \\ 0 & 1 & 8 & -5 \\ 0 & 0 & 1 & -\frac{21}{19} \end{array}\right] \xrightarrow{R2\to R2-8R3}$$

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & \frac{32}{19} \\ 0 & 1 & 0 & \frac{73}{19} \\ 0 & 0 & 1 & -\frac{21}{19} \end{array}\right]$$

Hence, the linear system has one solution: $\boxed{x_1 = 32/19,\ x_2 = 73/19 \text{ and } x_3 = -21/19}$.

N.B., You can always check your answer by re-substituting the solution you found into the original system.

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