COMMENTS ON SPAN - 9.26.2014

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Span, Span, Span, Span, Lovely Span, Wonderful Span[1].

Let V be a vector space (over $\mathbb{R}$). Let $X = \{\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k\}$ be a nonempty collection of vectors where each $\mathbf{v}_i \in V$ for $1 \leq i \leq k$. The **span of** X, denoted by Span(X) is the set of all linear combinations of the vectors in X. That is,

$$\text{Span}(X) = \{c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k \mid c_i \in \mathbb{R}, \mathbf{v}_i \in X \text{ where } 1 \leq i \leq k\}.$$

Recall that for any X as above, $\boxed{\text{Span}(X) \text{ is a subspace of } V}$. So the span is a fundamental and simple way to construct a subspace of V.

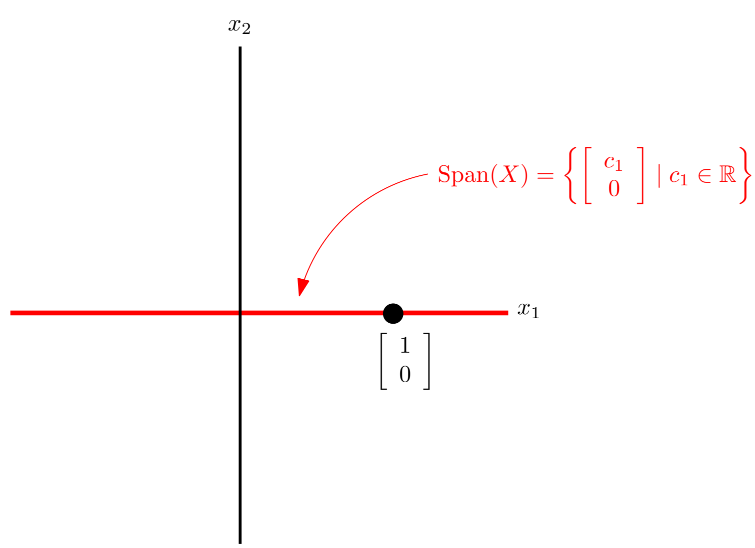
Example: Let $X = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ in $\mathbb{R}^2$.

(1) Find Span(X).
(2) Graph Span(X).

Solution for (1):

$$\text{Span}(X) = \text{Span}\left(\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}\right) = \left\{ c_1 \begin{bmatrix} 1 \\ 0 \end{bmatrix} \mid c_1 \in \mathbb{R} \right\} = \left\{ \begin{bmatrix} c_1 \\ 0 \end{bmatrix} \mid c_1 \in \mathbb{R} \right\}.$$

Solution for (2):



[1] A group of Vikings in a parallel universe to the Monty Python sketch "Spam".

N.B. Geometrically this is the x_1-axis since any point on the x_1-axis is of the form $(c_1, 0)$ where $c_1 \in \mathbb{R}$ so it corresponds to the vector $\begin{bmatrix} c_1 \\ 0 \end{bmatrix}$ where $c_1 \in \mathbb{R}$. Notice of course the vector $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is in Span(X) (more generally any vector in X will be in Span(X) since they are indeed linear combinations of vectors from X). We also know Span$\left(\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right\}\right)$ is a subspace of $\mathbb{R}^2$.

Example 2: In $\mathbb{R}[x]$ calculate Span$(\{1, x, x^2, x^3\})$:
Solution:

$$\begin{aligned} \text{Span}(\{1, x, x^2, x^3\}) &= \{c_0 \cdot 1 + c_1 \cdot x + c_2 \cdot x^2 + c_3 \cdot x^3 \mid c_1, c_2, c_3, c_4 \in \mathbb{R}\} \\ &= \{c_0 + c_1 x + c_2 x^2 + c_3 x^3 \mid c_1, c_2, c_3, c_4 \in \mathbb{R}\} \\ &= P_3 \end{aligned}$$

Recall, P_3 is the subspace of all polynomials in x with real coefficients of degree at most 3. Therefore, we were able to recognize P_3 as the span of some basic polynomials in $\mathbb{R}[x]$. This is no coincidence. In general, a subspace W of a vector space V can always be recognized as a span of a collection of vectors in V (not necessarily unique!). In other words:

Spans are subspaces and subspaces are spans.

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