EXAM 1

Score: ________ out of 100

Math 324 - Linear Algebra **Name:** key

Read all of the following information before starting the exam:

- You have 50 minutes to complete the exam.
- Show all work, clearly and in order, if you want to get full credit. Please make sure you read the directions for each problem. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Please $\boxed{\text{box/circle}}$ or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements.
- This test has 8 problems and is worth 100 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

1. Circle your answer for each of the following:

(a) True | False | A homogeneous linear system is always be consistent.

(b) True | False | A linear system may have exactly 5 solutions.

(c) True | False | If A is a 4×7 matrix and B is a 7×7 matrix, then BA exsits.

(d) True | False | If B has a column of zeros, then so does AB (if this product is defined).

(e) True | False | If A is an $n \times n$ matrix, then $\mathrm{tr}(5A) = 5\mathrm{tr}(A)$.

(f) True | False | If A is invertible, then so is A^{T}.

(g) True | False | The sum of two invertible matrix of the same size must be invertible.

$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

(h) True | False | Every elementary matrix is invertible.

(i) True | False | If A is an $n \times n$ matrix, then $\det(A^2) = (\det(A))^2$.

(j) If A is a 3×3 invertible matrix then $\det(3A) =$ | $3\det(A)$ | $9\det(A)$ | $27\det(A)$

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2. (a) Show that $\begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & 3 & -4 & 1 \\ -1 & -5 & -1 & 1 \end{bmatrix}$ has reduced row echelon (RREF) form $\begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

You must show each step of your work for full credit.

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 1 & 3 & -4 & 1 \\ -1 & -5 & -1 & 1 \end{bmatrix} \xrightarrow[R3 \to R3+R1]{R2 \to R2-R1} \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -7 & -1 \\ 0 & -3 & 2 & 3 \end{bmatrix} \xrightarrow{R3 \to R3+3R2} \begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -7 & -1 \\ 0 & 0 & -19 & 0 \end{bmatrix} \xrightarrow{R_3 \to \frac{-1}{19}R_3}$$

$$\begin{bmatrix} 1 & 2 & 3 & 2 \\ 0 & 1 & -7 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow[R_1 \to R_1 - 3R_3]{R_2 \to R_2 + 7R_3} \begin{bmatrix} 1 & 2 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \xrightarrow{R_1 \to R_1 - 2R_2} \begin{bmatrix} 1 & 0 & 0 & 4 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

(b) Use part (a) to find the solution of the system:

$$\begin{aligned} x_1 + 2x_2 + 3x_3 + 2x_4 &= 0 \\ x_1 + 3x_2 - 4x_3 + x_4 &= 0 \\ -x_1 - 5x_2 - x_3 + x_4 &= 0 \end{aligned}$$

$$\left[\begin{array}{cccc|c} 1 & 2 & 3 & 2 & 0 \\ 1 & 3 & -4 & 1 & 0 \\ -1 & -5 & -1 & 1 & 0 \end{array}\right] \xrightarrow{\text{RREF.}} \left[\begin{array}{cccc|c} 1 & 0 & 0 & 4 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{array}\right]$$

$$\left.\begin{aligned} x_1 + 4x_4 &= 0 \\ x_2 - x_4 &= 0 \\ x_3 &= 0 \end{aligned}\right\} \longrightarrow \boxed{\begin{aligned} x_1 &= -4t \\ x_2 &= t \\ x_3 &= 0 \\ x_4 &= t \end{aligned}} \quad \text{OR} \quad \boxed{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t \begin{bmatrix} -4 \\ 1 \\ 0 \\ 1 \end{bmatrix}}$$

3. Let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix}$$

Compute each of the following, or explain/show why it is NOT possible to compute.

(a) A^{T}

$$\begin{bmatrix} 1 & 0 \\ 1 & -1 \end{bmatrix}$$

(b) $A + 3B$

$$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} + \begin{bmatrix} 0 & 3 \\ 0 & -9 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 0 & -10 \end{bmatrix}$$

(c) AB

$$\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix} = \begin{bmatrix} 0 & 1-3 \\ 0 & 3 \end{bmatrix} = \begin{bmatrix} 0 & -2 \\ 0 & 3 \end{bmatrix}$$

(d) BA

$$\begin{bmatrix} 0 & 1 \\ 0 & -3 \end{bmatrix}\begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ 0 & 3 \end{bmatrix}$$

(e) $\mathrm{tr}(A)$ $= 1 + (-1) = \boxed{0}$

(f) $\det(A)$ $= \underbrace{(1)(-1)}_{\text{since } A \text{ is diagonal.}} = \boxed{-1}$

SOL 2: $ad - bc = (1)(-1) - (0)(1) = \boxed{-1}$

(g) A^{-1} $= \frac{1}{\det(A)}\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} = -\begin{bmatrix} -1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 0 & -1 \end{bmatrix}$

SOL 2: $[A \mid I] \longrightarrow \left[I \,\middle|\, \begin{matrix} 1 & 1 \\ 0 & -1 \end{matrix} \right]$ (work not shown here).

(h) $\det(B)$ $= (0)(-3) = \boxed{0}$

(i) B^{-1} does NOT exist since $\det(B) = 0$.

4. Let $A = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$

(a) Compute A^{-1}.

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 1 & 0 \\ 0 & -3 & 1 & 0 & 0 & 1 \end{array}\right] \xrightarrow{R2 \to R2+2R1} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & -3 & 1 & 0 & 0 & 1 \end{array}\right]$$

$$\xrightarrow{R3 \to R3+3R2} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 1 & 6 & 3 & 1 \end{array}\right]$$

$$\boxed{A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & 3 & 1 \end{bmatrix}}$$

(b) Use the A^{-1} computed in part (a) to solve the equation

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 6 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1+0+0 \\ 2-1+0 \\ 6-3+1 \end{bmatrix}$$

$$= \boxed{\begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix}}$$

5. Compute the following determinant: $\begin{vmatrix} 2 & 0 & 2 & -1 \\ 2 & 0 & 6 & -1 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{vmatrix} \xrightarrow{R2 \to R2-R1} \begin{vmatrix} 2 & 0 & 2 & -1 \\ 0 & 0 & 4 & 0 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 0 & 1 \end{vmatrix}$

$$\xrightarrow{R2 \leftrightarrow R3} -\begin{vmatrix} 2 & 0 & 2 & -1 \\ 0 & 3 & 0 & 3 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{vmatrix}$$

$$= -(2)(3)(4)(1)$$

$$= \boxed{-24}$$

6. Let $A = \begin{bmatrix} 3 & 1 & -4 \\ 0 & 1 & 3 \\ 0 & 0 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 5 & 0 & 0 \\ 3 & 1 & 0 \\ 4 & -5 & 1 \end{bmatrix}$

(a) $\det(A) = \boxed{3}$ (since triangular multiply diagonal entries)

(b) $\det(B) = \boxed{5}$

For the remaining parts you may not calculate the matrices involved, and should use your answers from parts (a) and (b):

(c) $\det(A^{-1}) = \boxed{1/3}$

(d) $\det(A^{T}) = \boxed{3}$

(e) $\det(A^2) = \boxed{9}$ $\det(A^2) = \det(AA) = \det(A) \cdot \det(A) = 3^2 = 9.$

(f) $\det(B^{-1}A^{-1}A^{T}A^2B) =$

$= \det(B^{-1})\det(A^{-1})\det(A^{T})\det(A^2)\det(B)$

$= (\frac{1}{5})(\frac{1}{3})(3)(9)5 = 9$

answer: $\boxed{9}$

For the following two problems 0 will be used to denote the zero matrix.

7. Show that if $A^5 - 2A^2 + A - I = \mathbf{0}$, then A^{-1} exists and $A^{-1} = A^4 - 2A + I$

SOL 1: $A^5 - 2A^2 + A = I$

$\underbrace{A(A^4 - 2A + I)}_{} = I$

So $A^{-1} = A^4 - 2A + I$. □

SOL 2: Claimed inverse: $A^4 - 2A + I$, so check!

$A(A^4 - 2A + I) = A^5 - 2A^2 + A$

$= I$

(since $A^5 - 2A^2 + A - I = 0$)

hence $A^{-1} = A^4 - 2A + I$ □.

8. Show that if $A^2 = \mathbf{0}$, then A is not invertible.

SOL 1: Direct proof:

Suppose $A^2 = 0$

$\det(A^2) = \det(\mathbf{0}) = 0$

$\downarrow$

$\det(A) \cdot \det(A) = 0$

so $\det(A) = 0$

$\Leftrightarrow$ A is not invertible.

SOL 2: by contradiction

Suppose $A^2 = 0$, but suppose A is invertible. Then

$A^{-1}A^2 = A^{-1}0$

$A = 0$

but then A is not invertible ← contradiction

here A is NOT invertible.