EXAM 3

Score: ________ out of 100

Math 324 - Linear Algebra **Name:** key

Read all of the following information before starting the exam:

- You have 50 minutes to complete the exam.
- Show all work, clearly and in order, if you want to get full credit. Please make sure you read the directions for each problem. I reserve the right to take off points if I cannot see how you arrived at your answer (even if your final answer is correct).
- Please $\boxed{\text{box/circle}}$ or otherwise indicate your final answers.
- Please keep your written answers brief; be clear and to the point. I will take points off for rambling and for incorrect or irrelevant statements.
- This test has 7 problems and is worth 100 points. It is your responsibility to make sure that you have all of the pages!
- Good luck!

1. Circle your answer for each of the following:

(a) True **False** P_n is isomorphic to $\mathbb{R}^n$. (P_n is isomorphic to $\mathbb{R}^{n+1}$)

(b) **True** False M_{mn} is isomorphic to $\mathbb{R}^{mn}$.

(c) **True** False There is a subspace of P_8 isomorphic to M_{22}. (Any 4 dimensional subspace will be isomorphic to M_{22})

(d) **True** False If V is a finite dimensional vector space and $T : V \to V$ is an isomorphism, then $\ker(T) = \{\mathbf{0}\}$.

Consider $T\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$

(e) True **False** Every linear transformation $T : M_{22} \to \mathbb{R}^4$ is an isomorphism. definitely not an isomorphism.

(f) **True** False If V and W are finite dimensional and isomorphic vector spaces, then $\dim(V) = \dim(W)$.

(g) True **False** $\begin{bmatrix} 1 & 0 \\ -1 & 2 \end{bmatrix}$ and $\begin{bmatrix} 3 & 1 \\ -1 & 0 \end{bmatrix}$ are similar matrices. (determinants are not equal. Hence, they cannot possibly be similar.)

(h) **True** False If A is a 3×3 matrix with three eigenvalues $\lambda_1 = 1$, $\lambda_2 = 5$ and $\lambda_3 = 15$, then A is diagonalizable.

(i) **True** False If A is a 3×3 matrix with three eigenvalues $\lambda_1 = 1$, $\lambda_2 = 5$ and $\lambda_3 = 15$, then A is invertible.

(j) **True** False A 3×3 matrix with real entries must always have at least one real eigenvalue.

↳ characteristic polynomial is a cubic which must always have at least one real root because complex roots come in conjugate pairs (also by the intermediate value thm.)

2. Find the eigenvalues of $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 4 \end{bmatrix}$.

Find roots of the characteristic equation

$$\det(A - \lambda I) = 0$$

$$\begin{vmatrix} 1-\lambda & 3 & 0 \\ 3 & 1-\lambda & 0 \\ 0 & 0 & 4-\lambda \end{vmatrix} = (+)(4-\lambda)\begin{vmatrix} 1-\lambda & 3 \\ 3 & 1-\lambda \end{vmatrix} = 0$$

$$= (4-\lambda)\big((1-\lambda)(1-\lambda) - 9\big) = 0$$

$$= (4-\lambda)\left(1 - 2\lambda + \lambda^2 - 9\right) = 0$$

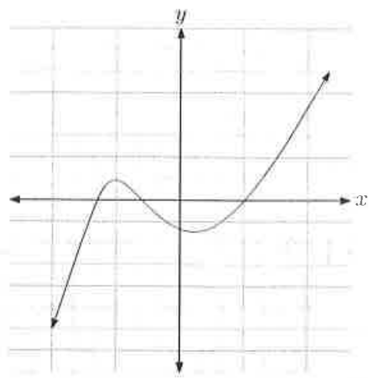
$$= (4-\lambda)\left(\lambda^2 - 2\lambda - 8\right) = 0$$

$$= (4-\lambda)(\lambda - 4)(\lambda + 2) = 0$$

$$\lambda = 4 \quad | \quad \lambda = 4 \quad | \quad \lambda = -2$$

eigenvalues of A: $\boxed{\begin{array}{l} \lambda = 4 \quad (\text{alg. multiplicity } 2) \\ \lambda = -2 \end{array}}$

3. Suppose the the graph of characteristic polynomial of an 3×3 matrix A is given below.



Circle your answer for each of the following:

(a) Is A invertible? | Yes | No | Not Enough Information |

Explain your choice:

The characteristic polynomial does <u>NOT</u> pass through the origin,
Hence, 0 is <u>NOT</u> an eigenvalue.
Therefore, A <u>is</u> invertible.

(b) Is A diagonalizable? | Yes | No | Not Enough Information |

Explain your choice:

The characteristic polynomial has 3 distinct roots.
Hence, A has 3 distinct eigenvalues.
Since A is 3×3, A <u>is</u> diagonalizable.

4. Suppose the characteristic polynomial of a square matrix A is:

$$p(\lambda) = \lambda(\lambda - 1)^2(\lambda + 2)^3$$

(a) Fill in the following table:

Eigenvalues of A	Algebraic Multiplicity	Possible Geometric Multiplicities
0	1	1
1	2	1, 2
-2	3	1, 2, 3

(b) Is A invertible? Circle your choice: | Yes | No | Not Enough Information |

(c) Is A diagonalizable? Circle your choice: | Yes | No | Not Enough Information |

(d) $\text{tr}(A) = \boxed{-4} = 0 + 1 + 1 + (-2) + (-2) + (-2)$ (sum of eigenvalues including multiplicities)

(e) $\det(A) = \boxed{0} = 0 \cdot 1 \cdot 1 \cdot (-2) \cdot (-2) \cdot (-2)$ (product of eigenvalues including multiplicities

↳ -OR- use the fact that A is <u>NOT</u> invertible to conclude $\det(A) = 0$.

(f) What is the size of the matrix A? $\boxed{6 \times 6}$

5. Let $A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$

(a) Find the eigenvalues of A.

SOLVE: $\det(A - \lambda I) = 0$

$$\begin{vmatrix} 1-\lambda & -1 \\ -1 & 1-\lambda \end{vmatrix} = 0$$

$$(1-\lambda)(1-\lambda) - (-1)(-1) = 0$$

$$1 - 2\lambda + \lambda^2 - 1 = 0$$

$$-2\lambda + \lambda^2 = 0$$

$$\lambda(-2+\lambda) = 0$$

$$\boxed{\lambda = 0 \quad \lambda = 2}$$

(b) For each eigenvalue from part (a), find a basis for the corresponding eigenspace.

For $\lambda = 0$ (The corresponding eigenspace is $\text{Null}(A - 0I) = \text{Null}(A)$.)

$$\left[\begin{array}{cc|c} 1-0 & -1 & 0 \\ -1 & 1-0 & 0 \end{array}\right] = \left[\begin{array}{cc|c} 1 & -1 & 0 \\ -1 & 1 & 0 \end{array}\right] \xrightarrow{R2 \to R2+R1} \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

FREE!

$x_1 = x_2 = s$

$x_2 = s$

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} s \\ s \end{bmatrix} = s\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Basis: $\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$

For $\lambda = 2$ (The corresponding eigenspace is $\text{Null}(A - 2I)$)

$$\left[\begin{array}{cc|c} 1-2 & -1 & 0 \\ -1 & 1-2 & 0 \end{array}\right] = \left[\begin{array}{cc|c} -1 & -1 & 0 \\ -1 & -1 & 0 \end{array}\right] \xrightarrow{R1 \to -R1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ -1 & -1 & 0 \end{array}\right] \xrightarrow{R2 \to R2+R1} \left[\begin{array}{cc|c} 1 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

FREE!

$$\left.\begin{array}{l} x_1 = -x_2 = -s \\ x_2 = s \end{array}\right\} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = s\begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

Basis: $\left\{ \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$

(c) Show that A is diagonalizable by stating a diagonalizing matrix P and diagonal matrix D so that $A = PDP^{-1}$. There is no need to check the last equality, just state what P and D are.

$$P = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

$$D = \begin{bmatrix} 0 & 0 \\ 0 & 2 \end{bmatrix}$$

6. Let $T : \mathbb{R}^3 \to P_2$ defined by

$$T\left(\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}\right) = (a_0 - a_1) + (a_1 - a_2)x + (a_2 - a_0)x^2.$$

(a) Find $[T]_{\mathcal{B}',\mathcal{B}}$ if $\mathcal{B}$ and $\mathcal{B}'$ are the standard bases (i.e., $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ and $\mathcal{B}' = \{1, x, x^2\}$)

$$[T]_{B',B} = \left[\left[T\left(\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}\right)\right]_{B'} \quad \left[T\left(\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}\right)\right]_{B'} \quad \left[T\left(\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}\right)\right]_{B'} \right]$$

$$= \left[\left[1 - 1x^2\right]_{B'} \quad \left[-1 + x\right]_{B'} \quad \left[-x + x^2\right]_{B'} \right]$$

$$= \boxed{\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}}$$

(b) Is T one-to-one (an injection)? Yes | No

Proof:

SOL 1: Notice that

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = 0$$

hence $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \in \ker(T)$

so, $\boxed{\text{T is NOT one to one}}$ because the kernel is not the trivial vector space

SOL 2:

$\vec{0}$ in P_2

$$\ker(T) = \left\{ \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \,\middle|\, T\left(\begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}\right) = 0 \right\}$$

$$= \left\{ \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \,\middle|\, (a_0 - a_1) + (a_1 - a_2)x + (a_2 - a_0)x^2 = 0 \text{ (for any } x) \right\}$$

$$= \left\{ \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \,\middle|\, \begin{array}{l} a_0 - a_1 = 0 \text{ AND} \\ a_1 - a_2 = 0 \text{ AND} \\ a_2 - a_0 = 0 \end{array} \right\}$$

$$= \left\{ \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix} \,\middle|\, a_0 = a_1 = a_2 \right\} = \left\{ s \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\} \neq \left\{ \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right\}$$

trivial vector space in $\mathbb{R}^3$

Hence $\boxed{\text{T is NOT one-to-one}}$

SOL 3:

$$T\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}\right) = 0 = T\left(\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}\right)$$

BUT $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \neq \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$

Hence $\boxed{\text{T is NOT one-to-one}}$

SOL 4 / 5

use the matrix above and show the Nullspace is NOT trivial

– OR –

show something like SOL 3. and get the same conclusion.

(c) Is T an isomorphism? Yes | No

Proof:

SOL 1:

From (b), since T is NOT one-to-one it is NOT an isomorphism.

SOL 2

Show the matrix in (a) is NOT invertible.

Many more solutions as well...

7. Prove that if A and B are similar matrices, then A^3 and B^3 are also similar matrices.

proof:

Suppose A and B are similar matrices.

There exists an invertible P such that

$$A = P^{-1} B P.$$

Now

$$A^3 = (P^{-1} B P)(P^{-1} B P)(P^{-1} B P)$$

$$= P^{-1} B^3 P$$

Hence, A^3 and B^3 are also similar matrices $\square$.