Math 324 - Induction Reminder
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Induction

Induction (The Principle of Mathematical Induction): Suppose $P(k)$ is a statement depending on the positive integer k and a is fixed positive integer. In order to prove

$$\text{“}P(k) \text{ is true for all positive integers } k \geq a,\text{”}$$

it is sufficient to prove:

1. **Base Case:** $P(a)$ is true, and
2. **Inductive Step:** $\forall n \geq a$, if $P(n)$ is true, then $P(n+1)$ is true. That is,
   (a) (**Induction Hypothesis**): Suppose $P(n)$ is true for SOME integer $n \geq a$.
   (b) Show: $P(n+1)$ is true.

Example 1: Show that for all positive integers k,

$$1 + 2 + \cdots + k = \frac{k(k+1)}{2}.$$

Proof. **Base Case:** (Show: $P(1)$ is true). The left hand side of the sum is 1 and the right hand side of the sum is 1(2)/2=1. Hence, $P(1)$ is true since 1=1.

Induction Step:
(a) **Induction Hypothesis:** Suppose $P(n)$ is true for SOME integer $n \geq 1$. That is, suppose

$$1 + 2 + \cdots + n = \frac{n(n+1)}{2}.$$

(b) (Show: $P(n+1)$ is true):

$$\begin{aligned} 1 + 2 + \cdots + n + (n+1) &= \frac{n(n+1)}{2} + (n+1) \text{ (by the induction hypothesis)} \\ &= \frac{n(n+1)}{2} + \frac{2(n+1)}{2} \\ &= \frac{n(n+1) + 2(n+1)}{2} \\ &= \frac{(n+1)(n+2)}{2}. \end{aligned}$$

Hence, $P(n+1)$ is true. $\square$

Strong Induction (The Second Principle of Mathematical Induction): Suppose $P(k)$ is a statement depending on the positive integer k, and a is fixed positive integer. In order to prove

"$P(k)$ is true for all positive integers $k \geq a$,"

it is sufficient to prove:

1. $P(a)$ is true, and
2. $\forall n \geq a$, if $P(a)$, $P(a+1)$, $P(a+2)$, $\ldots$, $P(n)$ are true, then $P(n+1)$ is true.

N.B. The base case may also start at an integer other than 1.
Example 2: Let $\{s_k\}$ be the sequence defined by

$$\begin{aligned} s_0 &= 0, \\ s_1 &= 1, \\ \text{and } \forall k \in \mathbb{Z} \text{ with } k \geq 2, s_k &= 3s_{k-1} - 2s_{k-2}. \end{aligned}$$

Show $\forall k \in \mathbb{Z}$ with $k \geq 0$, $s_k = 2^k - 1$.

Proof. **Base Case:** (Show: $P(0)$ is true). We are given $s_0 = 0$ and since $2^0 - 1 = 1 - 1 = 0$ we have $s_0 = 2^0 - 1$. Hence, $P(0)$ is true.

Induction Step:
(a) **Induction Hypothesis:** Suppose $P(0)$,$P(1)$,$P(2)$,$\ldots$,$P(n)$ are all true for SOME n.
(b) (Show: $P(n+1)$ is true).

$$\begin{aligned} s_{n+1} &= 3s_n - 2s_{n-1} \\ &= 3(2^n - 1) - 2(2^{n-1} - 1) \text{ (by the induction hypothesis)} \\ &= 3 \cdot 2^n - 3 - 2 \cdot 2^{n-1} + 2 \\ &= 3 \cdot 2^n - 2 \cdot 2^{n-1} - 1 \\ &= 3 \cdot 2^n - 2^n - 1 \\ &= (3-1) \cdot 2^n - 1 \\ &= (2) \cdot 2^n - 1 \\ &= 2^{n+1} - 1. \end{aligned}$$

Hence, $P(n+1)$ is true. $\square$