Group Homomorphisms. / 1st isomorphism Thm.

Q1 consider the mapping $\varphi : GL_2(\mathbb{R}) \to (\mathbb{R}^*, \cdot)$ defined by :

$$\varphi(A) = \det(A).$$

(i) Show that φ is a group homomorphism.

proof: φ is well defined since if $A \in GL_2(\mathbb{R})$ then by definition $\det(A) \neq 0$ and so $\det(A) \in \mathbb{R}^*$.

Now to show φ is a homomorphism :

Let $A, B \in GL_2(\mathbb{R})$.

$$\begin{aligned} \varphi(AB) &= \det(AB) \\ &= \det(A)\det(B) \quad \text{(property of determinants)} \\ &= \varphi(A)\varphi(B). \end{aligned}$$

Hence φ is a group homomorphism $\square$.

(ii) Find $\ker \varphi$

SOL: $\ker \varphi = \{ A \in GL_2(\mathbb{R}) \mid \varphi(A) = 1 \}$ identity element in $\mathbb{R}^*$

$\ker \varphi = \{ A \in GL_2(\mathbb{R}) \mid \det(A) = 1 \}$

$$\boxed{\ker \varphi = SL_2(\mathbb{R}).}$$

N.B. This result also shows $SL_2(\mathbb{R}) \triangleleft GL_2(\mathbb{R})$ which is an alternative to proving :

$\forall X \in GL_2(\mathbb{R})$, $X(SL_2(\mathbb{R}))X^{-1} \subseteq SL_2(\mathbb{R})$.

(iii) Find image $\varphi = \underbrace{\varphi(G)}$ ← Notation we used.

SOL: image $\varphi = \varphi(G) = \{ \varphi(A) \mid A \in GL_2(\mathbb{R}) \}$

$\varphi(G) = \{ \det(A) \mid A \in GL_2(\mathbb{R}) \}$

$\boxed{\varphi(G) = \mathbb{R}^*}$ ← Since for every nonzero real number y there is a matrix A with $\det(A) = y$.

(iv) use the 1st isomorphism thm to establish an isomorphism using the above.

SOL:

Recall, 1st isom thm says:

$$G/\ker\varphi \cong \varphi(G)$$

so

$$\boxed{GL_2(\mathbb{R})/SL_2(\mathbb{R}) \cong (\mathbb{R}^*, \cdot)}$$

(v) Is φ one-to-one (injective)?

SOL: $\boxed{\text{NO}}$ proof 1: $\ker\varphi = SL_2(\mathbb{R})$ from (b)

so the $\ker\varphi$ is NOT trivial (only containing the identity)

i.e., $\ker\varphi \neq \{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \}$

Hence, φ is NOT one-to-one. □

proof 2: Notice $\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}$ are different elements of $GL_2(\mathbb{R})$, but

$\varphi(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}) = \det(\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}) = 2$ } SAME

$\varphi(\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}) = \det(\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}) = 2$

which means φ is NOT one-to-one. □

(vi) Is φ onto (surjective)?

SOL: $\boxed{\text{YES}}$ proof 1 : see (iii) above

image $\varphi = \varphi(G) = \mathbb{R}^*$

codomain.

Hence φ is ONTO. $\square$

proof 2: Let $y \in \mathbb{R}^*$

(Show $\exists A \in GL_2(\mathbb{R})$ s.t. $\det(A) = y$)

consider $A = \begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix}$ (many others will work)

$\varphi(A) = \det(A) = \det\left(\begin{bmatrix} y & 0 \\ 0 & 1 \end{bmatrix}\right) = y.$

Hence, φ is ONTO. $\square$

(vii) Is φ bijective (one-to-one correspondence)

SOL: $\boxed{\text{NO}}$ It is not injective by (v) above.

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(Q2) Consider $\varphi : (\mathbb{R}^*, \cdot) \longrightarrow (\mathbb{R}^*, \cdot)$
defined by

$$\varphi(x) = |x|$$

(i) Show that φ is a group homomorphism

<u>proof</u>: φ is well defined since $|x| > 0$ if $x \neq 0$.
Thus $|x| \in \mathbb{R}^*$.

φ is a homomorphism:
Let $a, b \in \mathbb{R}^*$ (domain).

$$\varphi(ab) = |ab| = |a||b| = \varphi(a)\varphi(b). \quad \square.$$

(ii) Find $\ker \varphi$:

domain. identity in codomain.

$$\ker \varphi = \{ x \in \mathbb{R}^* \mid \varphi(x) = 1 \}$$
$$= \{ x \in \mathbb{R}^* \mid |x| = 1 \}$$
$$= \{ 1, -1 \} = \langle -1 \rangle$$

(iii) Find ~~ker~~ image $\varphi = \varphi(\mathbb{R}^*)$

$$\text{image } \varphi = \varphi(\mathbb{R}^*) = \{ \varphi(x) \mid x \in \mathbb{R}^* \}$$
$$= \{ |x| \mid x \in \mathbb{R}^* \}$$
$$= \mathbb{R}^+$$

(iv) What isomorphism does the 1st Isom. Thm. Establish?

$$(\mathbb{R}^*, \cdot)/\ker\varphi \cong \varphi(\mathbb{R}^*)$$

$$\boxed{(\mathbb{R}^*, \cdot)/\langle -1 \rangle \cong \mathbb{R}^+}$$

N.B.

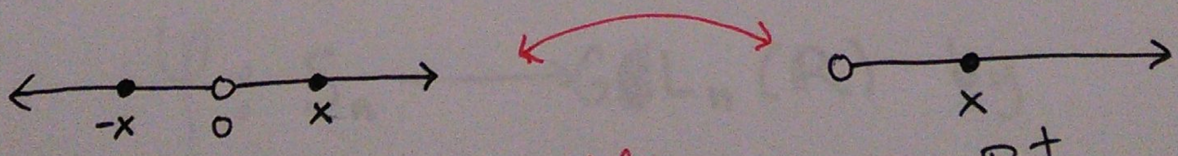
an element in the factor (quotient) group are <u>cosets</u>: if $x \in \mathbb{R}^*$ we can find a coset

$$x\langle -1 \rangle = \{x(1), x(-1)\}$$
$$= \{x, -x\}$$

So every nonzero real number is in one of these cosets. and the above says there is a correspondence between these cosets and the positive real numbers. do you see the correspondence?

$$\underbrace{x\langle -1 \rangle = \{x, -x\}}_{\text{element of the factor group } (\mathbb{R}^*, \cdot)/\langle -1 \rangle} \longleftrightarrow \underbrace{x}_{\text{element of } \mathbb{R}^+}$$

<u>Some Geometry</u>:



$\mathbb{R}^*$

The quotient action identifies $-x$ and x (since they are in the same coset.). Think of it as folding and unfolding

Q3 (Interesting, but more of a challenge).

Given a permutation $\sigma \in S_n$ we can associate a matrix: First consider σ in array form:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & n \\ \sigma(1) & \sigma(2) & \sigma(3) & \sigma(4) & \cdots & \sigma(n) \end{pmatrix}.$$

create the permutation matrix

$$P_\sigma = (p_{ij}) \in GL_n(\mathbb{R})$$

defined by

$$p_{ij} = \begin{cases} 1 & \text{if } \sigma(i) = j \\ 0 & \text{otherwise.} \end{cases}$$

e.g. 1 say $\sigma = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{pmatrix} \in S_3$. so $\sigma(1) = 2$, $\sigma(2) = 1$, $\sigma(3) = 3$

so we put a 1 in the (1,2)-entry, (2,1)-entry and (3,3)-entry)

$$P_\sigma = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Now define

$$\varphi : S_n \longrightarrow GL_n(\mathbb{R}) \text{ by}$$

$$\varphi(\sigma) = P_\sigma$$

(i) Is φ well defined?

proof: we need to show $P_\sigma \in GL_n(\mathbb{R})$.

$\boxed{\text{YES}}$

$\det(P_\sigma) = +1$ or -1

since we can interchange rows/cols to eventually get I and $\det(I) = 1$, the interchanges only introduce negations. (see Linear Algebra Notes.).

This means $\det(P_\sigma) \neq 0$.

N.B. This method shows that for a $\sigma \in S_n$, then $\det(P_\sigma) = +1$ or -1

so there are two types of permutations.

σ where $\det(P_\sigma) = +1$ and

σ where $\det(P_\sigma) = -1$

This is another way to define A_n:

Challenge to check: $A_n = \{ \sigma \in S_n \mid \det(P_\sigma) = +1 \}$

Harder Challenges:

(ii) Is φ a homomorphism?

(iii) Find $\ker \varphi$

(iv) Find $\varphi(S_n)$

We can actually compose maps:

$$S_n \xrightarrow{\text{find permutation matrix}} GL_n(\mathbb{R}) \xrightarrow{\text{take determinant}} \mathbb{R}^*$$

$$\sigma \longmapsto P_\sigma \longmapsto \det(P_\sigma)$$

Can you show this is a homomorphism?

i.e., composition is a homomorphism.

what is the kernel?

This is A_n, which shows

$A_n \triangleleft S_n$ (yet <u>another</u> way to show A_n is normal in S_n)