(1) Show that the set of 2×2 matrices with real entries is a group under matrix addition. The set builder notation for this is

$$M_2(\mathbb{R}) = \left\{ \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \,\middle|\, a_1, b_1, c_1, d_1 \in \mathbb{R} \right\}.$$

So Show $M_2(\mathbb{R})$ is a group under matrix addition.

$\boxed{\text{proof}}$ (i) Show closure: Let $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in M_2(\mathbb{R})$

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix} \in M_2(\mathbb{R})$$

(ii) Associativity: Let $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}, \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix} \in M_2(\mathbb{R})$

$$\begin{aligned}
\left(\begin{bmatrix} a_1 & b_1 \\ c_1 & a_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix}\right) + \begin{bmatrix} a_3 & b_3 \\ c_3 & a_3 \end{bmatrix} &= \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix} + \begin{bmatrix} a_3 & b_3 \\ c_3 & d_3 \end{bmatrix} \\
&= \begin{bmatrix} (a_1+a_2)+a_3 & (b_1+b_2)+b_3 \\ (c_1+c_2)+c_3 & (d_1+d_2)+d_3 \end{bmatrix} \\
&= \begin{bmatrix} a_1+(a_2+a_3) & b_1+(b_2+b_3) \\ c_1+(c_2+c_3) & d_1+(d_2+d_3) \end{bmatrix} \\
&= \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \left(\begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} + \begin{bmatrix} a_3 & b_3 \\ c_3 & a_3 \end{bmatrix}\right)
\end{aligned}$$

(iii) identity: $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ is the identity in $M_2(\mathbb{R})$.

indeed $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$

(iv) inverses: The inverse of $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ is $\begin{bmatrix} -a_1 & -b_1 \\ -c_1 & -d_1 \end{bmatrix}$

indeed $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} -a_1 & -b_1 \\ -c_1 & -d_1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} -a_1 & -b_1 \\ -c_1 & -d_1 \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$ $\square$

In general, $M_n(\mathbb{R})$ is a group under matrix addition.
What if we had entries not in $\mathbb{R}$? consider $\mathbb{Z}, \mathbb{Q}, \mathbb{C}, \mathbb{Z}_n$ etc.

② Is $M_2(\mathbb{R})$ abelian?

__yes!__ __proof__: Let $\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}, \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} \in M_2(\mathbb{R})$

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1+a_2 & b_1+b_2 \\ c_1+c_2 & d_1+d_2 \end{bmatrix}$$

$$= \begin{bmatrix} a_2+a_1 & b_2+b_1 \\ c_2+c_1 & d_2+d_1 \end{bmatrix} \quad \text{(since } \mathbb{R} \text{ is commutative under } +\text{)}$$

$$= \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} + \begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix}$$

$\square$

In general $M_n(\mathbb{R})$ is abelian. What if we had entries not in $\mathbb{R}$?

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④ Is the set $\{0, 1, 2, 3\}$ a group under multiplication modulo 4?

let's take a look at the multiplication table:

ab(mod 4)	0	1	2	3
0	0	0	0	0
1	0	1	2	3
2	0	2	0	2
3	0	3	2	1

It looks like 1 is our potential identity since $(a)(1) \bmod 4 = a$, but do you see any problems?

0 does not have an inverse because 1 does not appear in its corresponding row.

2 also does not have an inverse.

(1 and 3 do have inverses, but this is not enough).

__NO__, this is not a group. $\square$

NOTE: The operation here is closed (but there are problems as stated when showing we get a group)

(5) (a) construct a multiplication table for the group $U(15)$ (Cayley table)

$$U(15) = \{1, 2, 4, 7, 8, 11, 13, 14\}$$

recall, the operation is multiplication modulo 15:

ab (mod 15)	1	2	4	7	8	11	13	14
1	1	2	4	7	8	11	13	14
2	2	4	8	14	1	7	11	13
4	4	8	1	13	2	14	7	11
7	7	14	13	4	11	2	1	8
8	8	1	2	11	4	13	14	7
11	11	7	14	2	13	1	8	4
13	13	11	7	1	14	8	4	2
14	14	13	11	8	7	4	2	1

(b) Is $U(15)$ abelian?

we could prove this elementwise, but in general can you see how to show this looking at the multiplication table (cayley table)?

Notice the table is symmetric across the main diagonal (almost like a symmetric matrix from Linea Algebra.).

(c) Find 7^{-1}.

Notice in the table on the row corresponding to 7 there is an entry of 1 (the identity) ~~below~~ in the column corresponding to 13. So $7^{-1} = 13$.

indeed: $(7 \cdot 13) \pmod{15} \equiv 91 \pmod{15} \equiv 1 \pmod{15}$

(d) Find all elements of $U(15)$ with the property that $x^2 = 1$.

looking at the table again, on the main diagonal we see the products $x^2 \pmod{15}$ and 1 appears for elements 1, 4, 11 and 14.

<u>Solution</u>: $x = 1, 4, 11$ or 14.

Take a look at HW problem: Chapter 3 #26. can you always find an $x \in U(n)$ so that $x^2 = 1$ (and not considering the trivial solution $x = 1$).

The answer here should be short because you only need to exibit an x value.

(e) Find $|U(15)|$

Recall, $|U(15)|$ = order of $U(15)$ = # of elements = 8.

Take a look at these two extra problems #31 and #32 in Chapter 3. They tie together the above ideas into some nice results.