Chapter 5 - Permutation Groups.

Q1 suppose $\alpha = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 1 & 2 & 4 & 5 \end{pmatrix}$ $\beta = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 3 & 4 & 2 & 1 \end{pmatrix}$

(i) Write α and β in cycle notation as a product of disjoint cycles:

$$\alpha = (132)(4)(5) = (132)$$
$$\beta = (15)(234)$$

(ii) Find $\alpha\beta$, ~~and~~ $\beta\alpha$, α^2 and β^2. Write your answer as a product of disjoint cycles.

$$\alpha\beta = (132)(15)(234) = (1534)(2) = (1534)$$
$$\beta\alpha = (15)(234)(132) = (1425)(3) = (1425)$$
$$\alpha^2 = (132)(132) = (123)$$
$$\beta^2 = (15)(234)(15)(234) = (1)(243)(5) = (243)$$

(iii) Compute the orders:

$$|\alpha|, |\beta|, |\alpha\beta|, |\beta\alpha|, |\alpha^2|, |\beta^2|$$

Recall the order of an element $g \in G$ (where G is a group) is the smallest positive integer n such that $g^n = e$.

SOL 1: In S_n the order of the elements can be computed as: if $\sigma \in S_n$, then $|\sigma| = \text{lcm}$ (disjoint cycle lengths.).

Hence,

$$|\alpha| = \text{lcm}(3) = 3$$ (since $\alpha = (132)$ is a cycle of length 3)

Check: indeed $\alpha = (132) \neq e$
$\alpha^2 = (123) \neq e$
$\alpha^3 = (132)(123) = (1)(2)(3) = e.$

$$|\beta| = \text{lcm}(2,3) = 6$$
$$|\alpha\beta| = \text{lcm}(4) = 4$$
$$|\beta\alpha| = \text{lcm}(4) = 4$$

$|\alpha^2| = \text{lcm}(3) = 3.$

$|\beta^2| = \text{lcm}(3) = 3.$

SOL 2: you can always repeatedly multiply an element by itself until you get to the identity (as in the check step of computing $|\alpha|$ in SOL 1) but this takes a while ... especially if there are large cycle lengths!

(iv) Write α and β as the product of 2-cycles (transpositions)

$$\alpha = (132) = (12)(13)$$
$$\beta = (15)(234) = (15)(24)(23)$$

(v) Determine if α is EVEN or ODD
β is EVEN or ODD.

by (iv) α is the product of 2 2-cycles so
α is EVEN (this means $\alpha \in A_5$)

by (iv) β is the product of 3 2-cycles so
β is ODD

(vi) Find α^{-1} and β^{-1}

$\alpha^{-1} = (231)$ (Check: indeed $\alpha\alpha^{-1} = (132)(231) = (1)(2)(3) = e$)

$\beta^{-1} = (15)(432)$

(vii) Find $(\alpha\beta)^{-1}$, $(\beta\alpha)^{-1}$, $(\alpha^2)^{-1}$, $(\beta^2)^{-1}$

<u>SOL 1</u>:

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$$(\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1} \overset{\text{by (vi)}}{=} (15)(432)(231)$$
$$= (1435)(2)$$
$$= (1435)$$

you can do the same for the others ...

<u>SOL 2</u>: use answers from (ii)

$$(\alpha\beta)^{-1} = (4351)$$ (Note, this is the same as SOL 1: $(4351) = (1435)$)

$$(\beta\alpha)^{-1} = (5241)$$
$$(\alpha^2)^{-1} = (321)$$
$$(\beta^2)^{-1} = (342)$$

Q2 What are the possible disjoint cycle structures of S_6?

Let's denote a cycle of length n by $(\underline{n})$
so for example a cycle of thength 6 is denoted by $(\underline{6})$
an example of a cycle of length 6 is actually (132456)

Now the elements of S_6 can be written as the product of disjoint cycles. so what are the disjoint cycle structures of S_6?

$(\underline{6})$
$(\underline{5})(\underline{1})$
$(\underline{4})(\underline{2})$
$(\underline{4})(\underline{1})(\underline{1})$
$(\underline{3})(\underline{3})$
$(\underline{3})(\underline{2})(\underline{1})$
$(\underline{3})(\underline{1})(\underline{1})(\underline{1})$
$(\underline{2})(\underline{2})(\underline{2})$
$(\underline{2})(\underline{2})(\underline{1})(\underline{1})$
$(\underline{2})(\underline{1})(\underline{1})(\underline{1})(\underline{1})$
$(\underline{1})(\underline{1})(\underline{1})(\underline{1})(\underline{1})(\underline{1})$

Q3 What are the possible orders of the elements in S_6?

We can compute the orders using the disjoint cycle lengths by taking the lcm(disjoint cycle lengths) as we did in Q1 (iii). In Q2 we determined all the possible disjoint cycle structures of S_6 hence, The possible orders are

$\text{lcm}(6) = 6$
$\text{lcm}(5,1) = 5$
$\text{lcm}(4,2) = 4$
$\text{lcm}(4,1,1) = 4$
$\text{lcm}(3,3) = 3$
$\text{lcm}(3,2,1) = 6$
$\text{lcm}(3,1,1,1) = 3$
$\text{lcm}(2,2,2) = 2$
$\text{lcm}(2,1,1,1,1) = 2$
$\text{lcm}(1,1,1,1,1,1) = 1$

So the possible orders are $1, 2, 3, 4, 5, 6$ in S_6.

(Q4) From Q3 it is tempting to ~~conjecture~~/ ask the following

Are the orders in S_n always just $1, 2, \ldots, n$?

$\boxed{\text{no}}$, consider $\sigma = (12)(345)$ in S_5
$|\sigma| = \text{lcm}(2,3) = 6$.

<u>Challenge question</u>: How big can $|\sigma|$ be if $\sigma \in S_n$?

Q5 Is the cycle $(a_1 a_2 \cdots a_n)$ EVEN or ODD?

we can write $(a_1 a_2 \cdots a_n)$ as a product of 2-cycles:

$$(a_1 a_2 \cdots a_n) = \underbrace{(a_1 a_n)(a_1 a_{n-1}) \cdots (a_1 a_2)}$$

Is the total number of 2-cycles Even or odd?

The answer depends on n:

if n is EVEN then $(a_1 a_2 \cdots a_n)$ is ODD
if n is ODD then $(a_1 a_2 \cdots a_n)$ is EVEN

example:

$$\underbrace{(2\,4\,5\,6\,3)}_{\text{length is 5 (odd number)}} = \underbrace{(2\,3)(2\,6)(2\,5)(2\,4)}_{\text{4 2-cycles, so the permutation is EVEN}}$$

$$\underbrace{(1\,2\,4\,5)}_{\text{length is 4 (even number)}} = \underbrace{(1\,5)(1\,4)(1\,2)}_{\text{3 2-cycles, so the permutation is ODD}}$$

(Q6) What are the possible disjoint cycle structures of A_6?

using the idea of (Q5) and the work of (Q2) we can determine which cycle types are in A_6:

$(\underline{6})$ is ODD (a 6 cycle becomes 5 2-cycles)
$(\underline{5})(\underline{1})$ is EVEN. (a 5 cycle becomes 4 2-cycles)
$(\underline{4})(\underline{2})$ is EVEN. (4-cycle is 3 2-cycles and the extra 2-cycle makes 4 2-cycles.)

$(\underline{4})(\underline{1})(\underline{1})$ is ODD (4-cycle is 3 2-cycles.)
$(\underline{3})(\underline{3})$ is EVEN (3-cycle is 2 2-cycles so a total of 4 2-cycles)
$(\underline{3})(\underline{2})(\underline{1})$ is ODD (3-cycle is 2 2-cycles total of 3 2-cycles)
$(\underline{3})(\underline{1})(\underline{1})(\underline{1})$ EVEN

$(\underline{2})(\underline{2})(\underline{2})$ ODD
$(\underline{2})(\underline{2})(\underline{1})(\underline{1})$ EVEN
$(\underline{2})(\underline{1})(\underline{1})(\underline{1})(\underline{1})$ ODD
$(\underline{1})(\underline{1})(\underline{1})(\underline{1})(\underline{1})(\underline{1})$ EVEN

So the only elements in A_6 are the EVEN elements above. we can actually further simplify if you ignore 1-cycles:

$(\underline{5})$
$(\underline{4})(\underline{2})$
$(\underline{3})(\underline{3})$
$(\underline{3})$
$(\underline{2})(\underline{2})$
$(\underline{1}) = e = \varepsilon$ (identity)