Q1 (a) Find all the left cosets of $H = \langle (12) \rangle$ in S_3.

SOL: Recall $S_3 = \{(1), (12), (13), (23), (123), (132)\}$

Here $H = \langle (12) \rangle = \{(12), (1)\}$

$$(12)^2 = (12)(12) = (1)(2) = e = (1)$$

Now, the left cosets can be computed:

$$(1)H = \{(1)(12), (1)(1)\} = \{(12), (1)\} = H = (12)H$$
$$(13)H = \{(13)(12), (13)(1)\} = \{(123), (13)\} = (123)H$$
$$(23)H = \{(23)(12), (23)(1)\} = \{(132), (23)\} = (132)H$$

Note: to get these equivalent cosets we are using the property: $\{aH = bH \leftrightarrow a \in bH\}$

This cuts our work down.

So there are three distinct left cosets.

(b) Can we determine the number of distinct left (or right) cosets without actually finding all of them?

(But we still need to do it the long way if asked to FIND the cosets)

SOL: YES : once you know the order of the group G AND the order of the subgroup H we can compute this

Recall, $|G:H|$ = index of H in G = # of distinct left (or right) cosets.

Lagrange's Thm: $|G:H| = \dfrac{|G|}{|H|}$

So we can confirm our work in (a) that shows there are 3 distinct left cosets of H in S_3. : $|S_3 : H| = \dfrac{|S_3|}{|H|} = \dfrac{3!}{2} = \dfrac{6}{2} = 3$

(Q2) How many distinct left cosets of $H = $ [redacted] $\langle s_1 \rangle$ in D_{20} are there if s_1 [redacted] is the reflection that fixes vertex 1?

SOL: $\left| D_{20} : H \right| = \dfrac{|D_{20}|}{|H|} = \dfrac{40}{2} = \boxed{20}$

$|D_{20}| = 40$ (20 rotations, 20 reflections } symmetries of regular 20-gon)

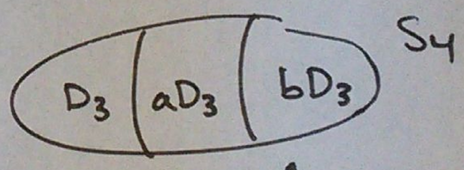
$|H| = 2$ since $H = \{ s_1, s_1^2 \} = \{ s_1, e \}$

all reflections have order 2

(Q3) How many distinct [redacted] right cosets of D_4 in S_4 are there?

SOL: $\left| S_4 : D_4 \right| = \dfrac{|S_4|}{|D_4|} = \dfrac{4!}{2 \cdot 4} = \dfrac{4 \cdot 3 \cdot 2 \cdot 1}{4 \cdot 2} = \boxed{3}$

Challenge: can you find them? This was a group HW problem



can you find a and b?

(there are many answers).

(Q4) How many distinct left (or right cosets) of $\langle 8 \rangle$ in $\mathbb{Z}_{24}$ are there?

SOL: $\langle 8 \rangle = \{ 0, 8, 16 \}$ so $|\langle 8 \rangle| = 3$.

$|\mathbb{Z}_{24}| = 24$.

Now

$\left| \mathbb{Z}_{24} : \langle 8 \rangle \right| = \dfrac{|\mathbb{Z}_{24}|}{|\langle 8 \rangle|} = \dfrac{24}{3} = \boxed{8}$

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Challenge : can you find them ?

Q5 How many distinct left (or right cosets) of $\langle 3 \rangle$ in $\mathbb{Z}_{24}$ are there ?

SOL :

$\langle 3 \rangle = \{0, 3, 6, 9, 12, 15, 18, 21\}$ so $|\langle 3 \rangle| = 8$

(NOTE: alternatively we can calculate $|\langle 3 \rangle| = |3| = 8$)

$|\mathbb{Z}_{24}| = 24$

Now

$$|\mathbb{Z}_{24} : \langle 3 \rangle| = \frac{|\mathbb{Z}_{24}|}{|\langle 3 \rangle|} = \frac{24}{8} = \boxed{3}$$

Challenge : can you find them ?

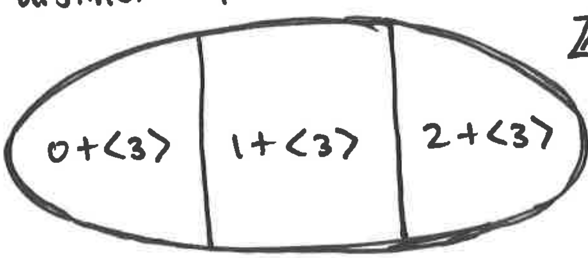
$$\begin{aligned}
0 + \langle 3 \rangle = \{0, 3, 6, 9, 12, 15, 18, 21\} &= 3 + \langle 3 \rangle \\
&= 6 + \langle 3 \rangle \\
&= 9 + \langle 3 \rangle \\
&= 12 + \langle 3 \rangle \\
&= 15 + \langle 3 \rangle \\
&= 18 + \langle 3 \rangle \\
&= 21 + \langle 3 \rangle
\end{aligned}$$

$$\begin{aligned}
1 + \langle 3 \rangle = \{1, 4, 7, 10, 13, 16, 19, 22\} &= 1 + \langle 3 \rangle \\
& \quad 4 + \langle 3 \rangle \\
& \quad 7 + \langle 3 \rangle \\
& \quad 10 + \langle 3 \rangle \\
& \quad 13 + \langle 3 \rangle \\
& \quad 16 + \langle 3 \rangle \\
& \quad 19 + \langle 3 \rangle \\
& \quad 22 + \langle 3 \rangle
\end{aligned}$$

$$\begin{aligned}
2 + \langle 3 \rangle = \{2, 5, 8, 11, 14, 17, 20, 23\} &= 2 + \langle 3 \rangle \\
&\ \ \vdots \\
&= 23 + \langle 3 \rangle
\end{aligned}$$

So these are the 3 distinct left cosets

All this is from the same property in Q1.



$\mathbb{Z}_{24}$

Q6 Suppose G is a group and $H \leq G$. Let $g \in G$.

(a) If $|G| = 8$, what are the possible orders of H?
i.e., what can $|H|$ be?

__SOL:__ By Lagrange's Thm $|H|$ must divide $|G|$. So and divisar of 8 can be the order of H.

That is, $|H|$ __could__ be 1, 2, 4 or 8

(b) If $|G| = 8$, what are the possible orders of g?
i.e., what can $|g|$ be?

__SOL:__ By Lagrange's thm (consequence of)
$|g|$ must also divide $|G|$. Hence

$|g|$ could be 1, 2, 4 or 8.

(c) If $|G| = 8$ and $|g| = 2$ does G have a subgroup of order 2?

__SOL:__ __yes__: $\langle g \rangle$ is always a subgroup __and__
$|\langle g \rangle| = |g| = 2$
so $\langle g \rangle$ is a subgroup of order 2
(in fact $\langle g \rangle = \{e, g\}$)

Q7 Let G be a group, $H \leq G$ AND $g \in G$.

True / False :

(a) If $|G| = 8$, then $|H|$ could be 5. FALSE (Lagrange's Thm)

(b) If $|G| = 10$, then $|H|$ could be 5. True (")

(c) If $|G| = 12$, then G MUST have a subgroup of order 6.

False

Further NOTE: we also proved A_4 has no subgroup of order 6.

Lagrange's Thm in general does not have a converse, but there are special cases... see next question.

(d) If $|G| = 12$, ~~[illegible]~~ AND suppose G is cyclic, then G MUST have a subgroup of order 6.

True

(Fundamental Thm of cyclic groups.)

(e) If $|G| = 12$, then G must have an element of order 6.

False

(Same idea as (c) above)

Q8 Recall $\mu_n \leq \mathbb{T} \leq \mathbb{C}^*$

μ_n: nth roots of unity

$\mathbb{T}$: circle group.

$\mathbb{C}^*$: all complex numbers (except 0) under multiplication.

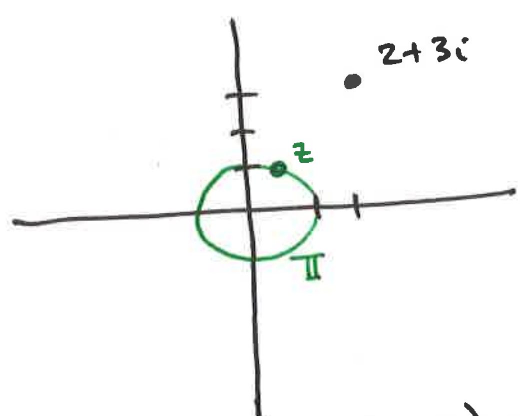
(The set definitions were: $\mu_n = \{ z \in \mathbb{C}^* \mid z^n = 1 \}$

$\mathbb{T} = \{ z \in \mathbb{C}^* \mid |z| = 1 \}$)

Can you geometrically describe the coset $(2+3i)\mathbb{T}$?

$$(2+3i)\mathbb{T} = \{ \underbrace{(2+3i)z}_{w} \mid z \in \mathbb{T} \}$$

so the collection of $w \in \mathbb{C}^*$ such that $w = (2+3i)z$ ~~and~~ where $z \in \mathbb{T}$



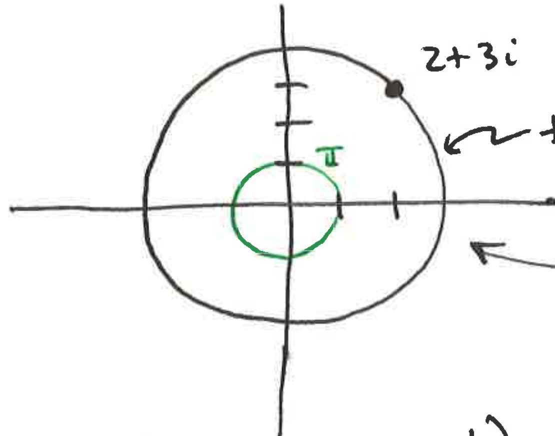
what is the modulus (absolute value) of w?

$$|w| = |(2+3i)z|$$
$$= |2+3i||z|$$
$$= |2+3i| \cdot 1$$
$$= \sqrt{2^2+3^2}$$
$$= \sqrt{4+9}$$
$$= \sqrt{13}$$

So w is the collection of all complex numbers whose distance to the origin (which is the modulus (absolute value)) is $\sqrt{13}$

i.e.,

$$|w| = |2+3i|$$

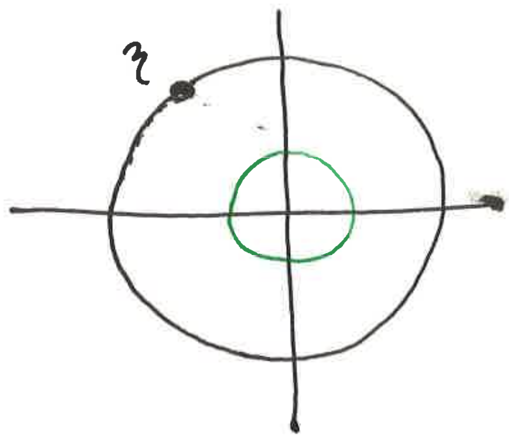


this is the collection of w.

i.e., this circle is the left coset $(2+3i)\mathbb{T}$

IN GENERAL: (this gets really neat)

Notice that all nonzero complex numbers z are on some circle centered at the origin. These circles are the left cosets $z\mathbb{T}$



all these circles are disjoint and nonempty and so they partition $\mathbb{C}^*$. This is consistent with what we know: The left (right) cosets partition the group.

This fact will be very useful (soon).

~~There is some hidden notion of equivalence relation~~

(Q9) Let G be a group with $|G| = p^2$, where p is prime.

Prove that either G is cyclic <u>OR</u> $g^p = e$ for all $g \in G$.

<u>SOL</u>: By Lagrange's Thm elements in g must have orders that divide p^2. i.e., $|g|$ is 1 or p or p^2.

~~Also assume $g \neq e$ (identity) so $|g| \neq 1$.~~

<u>Case 1</u>: $|g| = p^2 \to g$ generates G so $\langle g \rangle = G$ ✓ hence <u>G is cyclic</u>.

<u>Case 2</u>: $|g| = p. \to \underline{g^p = e}.$ ✓

<u>Case 3</u>: $|g| = 1 \to g = e$ but we also have $\underline{g^p = e^p = e}$ ✓

In each case we ended up with one of the two desired conclusions, so we are good $\square$.