# Direct Products.

**Q1** The group $S_3 \times \mathbb{Z}_2$ is isomorphic to one of the following groups:

$$\mathbb{Z}_{12}, \quad \mathbb{Z}_6 \times \mathbb{Z}_2, \quad A_4, \quad D_6$$

Determine which one by elimination.

**SOL**:

- $\mathbb{Z}_{12}$ is abelian and $S_3 \times \mathbb{Z}_2$ is not since $S_3$ is not abelian. So $\mathbb{Z}_{13}$ is out.

- Similarly, $\mathbb{Z}_6 \times \mathbb{Z}_2$ is out for the same reason.

- To eliminate $A_4$ we can look at the order of some elements. In $A_4$ we only have

  1 element of order 1 (the identity).
  3 elements of order 2
  8 elements of order 3

  there are several ways to show these orders do not match those in $S_3 \times \mathbb{Z}_2$ but here is one:

  consider the element $((123), 1) \in S_3 \times \mathbb{Z}_2$

  its order is $$|((123),1)| = \text{lcm}(\underbrace{|(123)|}_{\text{in } S_3}, \underbrace{|1|}_{\text{in } \mathbb{Z}_2})$$

  $$= \text{lcm}(3,2)$$

  $$= 6$$

  but $A_4$ has no elements of order 6, so $S_3 \times \mathbb{Z}_2 \ncong A_4$

- This leaves $D_6$ as the only option. Indeed $\boxed{D_6 \cong S_3 \times \mathbb{Z}_2}$ $\square$

**Q2** Consider the group $\mathbb{Z}_4 \times \mathbb{Z}_2$

(i) Calculate $|\mathbb{Z}_4 \times \mathbb{Z}_2|$

SOL: The order of a direct product is the product of ~~the~~ the ~~groups~~ orders of the groups that construct it:

$$|\mathbb{Z}_4 \times \mathbb{Z}_2| = |\mathbb{Z}_4||\mathbb{Z}_2| = 4 \cdot 2 = \boxed{8}$$

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(ii) Find the order of the element $(1,0) \in \mathbb{Z}_4 \times \mathbb{Z}_2$

SOL 1: $$|(1,0)| = \operatorname{lcm}(\underbrace{|1|}_{\text{in } \mathbb{Z}_4}, \underbrace{|0|}_{\text{in } \mathbb{Z}_2}) = \operatorname{lcm}(4,1) = \boxed{4}$$

SOL 2: $\langle (1,0) \rangle = \{(0,0), (1,0), (2,0), (3,0)\}$

hence $|(1,0)| = |\langle (1,0) \rangle| = \boxed{4.}$

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(iii) Is $\mathbb{Z}_4 \times \mathbb{Z}_2$ abelian?

SOL: yes! Recall, $G_1 \times G_2 \times \cdots \times G_n$ is abelian $\Longleftrightarrow \forall i, G_i$ is abelian.

So since both $\mathbb{Z}_4$ AND $\mathbb{Z}_2$ are abelian

$\boxed{\mathbb{Z}_4 \times \mathbb{Z}_2 \text{ is also abelian.}}$

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(iv) Is $\mathbb{Z}_4 \times \mathbb{Z}_2$ cyclic?

SOL: No, since $\mathbb{Z}_4$ and $\mathbb{Z}_2$ are cyclic it is NOT enough, because $\gcd(4,2) \neq 1$.

so $\mathbb{Z}_4 \times \mathbb{Z}_2 \not\cong \mathbb{Z}_8$ and it is NOT cyclic.

(v) Find a subgroup of $\mathbb{Z}_4 \times \mathbb{Z}_2$ that is NOT of the form $H \times K$, where $H \leq \mathbb{Z}_4$ and $K \leq \mathbb{Z}_2$.

Sol: consider the cyclic subgroup:

$$\langle (1,1) \rangle = \{ (0,0), (1,1), (2,0), (3,1) \}$$

The subgroups of the form $H \times K$, where $H \leq \mathbb{Z}_4$, $K \leq \mathbb{Z}_2$ are as follows:

| $H$ | $K$ | $H \times K$ |
|---|---|---|
| $\{0\}$ | $\{0\}$ | $\{0\} \times \{0\} = \{(0,0)\}$ |
| $\langle 2 \rangle$ | $\mathbb{Z}_2$ | $\{0\} \times \mathbb{Z}_2 = \{(0,0), (0,1)\}$ |
| $\mathbb{Z}_4$ | | $\langle 2 \rangle \times \{0\} = \{(0,0), (2,0)\}$ |
| | | $\langle 2 \rangle \times \mathbb{Z}_2 = \{(0,0), (0,1), (2,0), (2,1)\}$ |
| | | $\mathbb{Z}_4 \times \{0\} = \{(0,0), (1,0), (2,0), (3,0)\}$ |
| | | $\mathbb{Z}_4 \times \mathbb{Z}_2 = \{(0,0), (1,0), (2,0), (3,0), (0,1), (1,1), (2,1), (3,1)\}$ |

Notice that $\langle (1,1) \rangle$ is NONE of these.

Q3 Consider $\mathbb{Z}_{14} \times \mathbb{Z}_{15}$

(i) prove or disprove $\mathbb{Z}_{14} \times \mathbb{Z}_{15} \cong \mathbb{Z}_{210}$

~~proof.~~ [illegible]
since $\gcd(14, 15) = 1 \rightarrow \mathbb{Z}_{14} \times \mathbb{Z}_{15} \cong \mathbb{Z}_{14 \cdot 15} = \mathbb{Z}_{210}$.

(ii) prove or disprove $\mathbb{Z}_{14} \times \mathbb{Z}_{15}$ is cyclic.

proof 1: by (i) $\mathbb{Z}_{14} \times \mathbb{Z}_{15} \cong \mathbb{Z}_{210}$ which shows it is cyclic!

proof 2: since $\mathbb{Z}_{14}$ and $\mathbb{Z}_{15}$ are cyclic AND $\gcd(14, 15) = 1$

$\downarrow$

$\mathbb{Z}_{14} \times \mathbb{Z}_{15}$ is cyclic.

proof 3: find a generator ...

Q4 Recall $M_{22}(\mathbb{R})$ is the group of all real $2 \times 2$ matrices under addition.
Let $N = \mathbb{R} \times \mathbb{R} \times \mathbb{R} \times \mathbb{R} = \mathbb{R}^4$ be the collection of vectors in $\mathbb{R}^4$ as a group under component wise addition.
prove that $M_{22}(\mathbb{R}) \cong \mathbb{R}^4$.

proof: Let $\varphi : M_{22}(\mathbb{R}) \longrightarrow \mathbb{R}^4$ be defined by

$$\varphi\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right) = (a, b, c, d).$$

$\varphi$ is an isomorphism. check!

Q5 prove that for any groups $G$ and $H$

$$G \times H \cong H \times G.$$

proof: Let $\varphi : G \times H \longrightarrow H \times G$ be defined by $\varphi((g,h)) = (h,g)$.

$\varphi$ is an isomorphism. check!

Q6 Let $(a,b) \in \mathbb{Z}_m \times \mathbb{Z}_n$. Prove that $|(a,b)|$ divides $\operatorname{lcm}(m,n)$.

Hint: use the fact that $|(a,b)| = \operatorname{lcm}(|a|,|b|)$ and Lagrange's Thm.