Answer these questions in your Blue Book (this test paper will not *be graded). Make sure to start each question on a new page. No calculators, cell phones, or anything else with a battery. Make sure to read all directions and show all your work. Answers missing work will lose points. If you use u-substitution anywhere, you must clearly indicate your u. For questions 5 and 6, if you don't sketch regions it's almost impossible to give you any partial credit. Good Luck.*

1. For this question $f(x) = \dfrac{x^3}{x-1}$.

   Most of this question has been done for you, you are to answer the questions in bold. If your answer to any question is none, make sure you write 'none.'

   - Natural Domain: $(-\infty, 1) \cup (1, \infty)$
   - Intercepts: $(0, 0)$

   (a) Asymptotes

   $\lim_{x\to\infty} \frac{x^3}{x-1} = \infty$ and $\lim_{x\to-\infty} \frac{x^3}{x-1} = \infty$

   $\lim_{x\to 1^-} \frac{x^3}{x-1} = -\infty$ and $\lim_{x\to 1^+} \frac{x^3}{x-1} = +\infty$

   (i) **Find All Vertical Asymptotes of $f(x)$**

   (ii) **Find All Horizontal Asymptotes of $f(x)$**

   (b) $f'(x) = \dfrac{(x^2)(2x-3)}{(x-1)^2}$

   Where is $f(x)$ increasing and decreasing? What are the local maxima and minima of $f(x)$?

   - $f''(x) = \dfrac{2x^3 - 6x^2 + 6x}{(x-1)^3}$

     $f(x)$ is Concave Up on $(-\infty, 0) \cup (1, \infty)$ Point(s) of Inflection: $(0, 0)$

     $f(x)$ is Concave Down on $(0, 1)$

   (c) **Use all the information in this question to sketch the graph of $f(x) = \dfrac{x^3}{x-1}$**

There are More Questions On The Back!!!

2. Suppose $0 \le x \le 10$, at which point(s) on the curve $y = x^3 - 6x^2 - 2x + 5$ does the tangent line have the smallest slope?

3. Evaluate the following integrals:

   (a) $\displaystyle\int \cos(x)\Big(\sin(x)\Big)^{1/3}\, dx$

   (b) $\displaystyle\int (1 - x^2)^2\, dx$

4. (a) Set up but do not evaluate $\displaystyle\int_0^2 5x^2\, dx$ using the limit definition of the integral (as the limit of Riemann sums).

   (b) Evaluate $\displaystyle\int_0^3 x\, dx$ (using any correct method).

   (c) Evaluate $\displaystyle\int_0^{\sqrt{8}} 3x\sqrt{1 + x^2}\, dx$ (using any correct method).

5. Set up but do not evaluate an area of the region bounded by $y = x^2 + 3$, $y = 4x$.

6. The region bounded by $y = \sqrt{x}$, $x = 4$ and the x-axis is rotated around the line $x = -1$. Set up but do not evaluate an integral for the volume of this solid.

① (a) (i) $x = 1$

(ii) None

(b) find the critical numbers of f:

$f'(x) = 0$

$$\frac{x^2(2x-3)}{(x-1)^2} = 0$$

$x^2(2x-3) = 0$

$x = 0$, $x = 3/2$

$f'(x)$ is undefined when

$(x-1)^2 = 0$

$x = 1$

(not in the domain of f anyway)

now let's do a sign analysis. Plot the domain line and mark critical numbers.

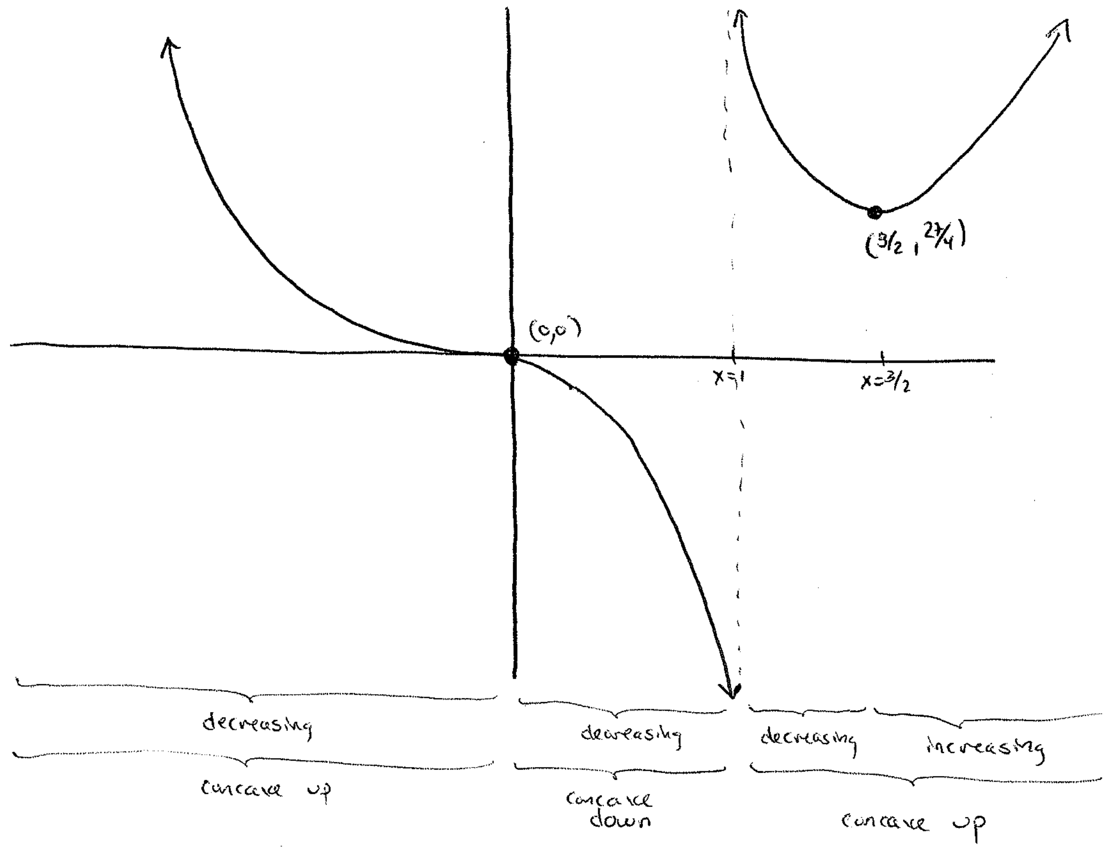
f'	$-$	$-$	$-$	$+$
	$x=0$	$x=1$	$x=3/2$	
intervals :	$(-\infty, 0)$	$(0, 1)$	$(1, 3/2)$	$(3/2, \infty)$
f :	decreasing	decreasing	decreasing	increasing

local minimum when $x = 3/2$

so $f(3/2) = \dfrac{(3/2)^3}{(3/2)-1} = \dfrac{27/8}{1/2} = \dfrac{27}{4}$

f is decreasing on $(-\infty, 0) \cup (0,1) \cup (1, 3/2)$
f is increasing on $(3/2, \infty)$
f has no local maxima
f has a local minima at the point $(3/2, 27/4)$

(c)



(0,0)
(3/2 , 27/4)
x=1
x=3/2
decreasing
concave up
decreasing
concave down
decreasing
increasing
concave up

(2) we start with the curve $y = x^3 - 6x^2 - 2x + 5$ over $0 \le x \le 10$.

The slope at a given x value is represented by the derivative of y (slope of the tangent line at x)

$$y' = 3x^2 - 12x - 2$$

now we want to find points which have the <u>smallest</u> slope. So we are finding the absolute minimum of the function

$$g(x) = 3x^2 - 12x - 2.$$

To find the absolute minimum we use the closed interval method:

find the critical numbers of g:

$$g'(x) = 6x - 12$$

$g'(x) = 0$
$6x - 12 = 0$
$x = \frac{12}{6} = 2$

$g'(x)$ is undefined ... never! $g'(x)$ is a polynomial.

So we only have one critical number $x = 2$.
now evaluate g at the critical number(s) and endpoints

$$g(0) = 3(0)^2 - 12(0) - 2 = -2$$
$$g(2) = 3(2)^2 - 12(2) - 2 = -14 \leftarrow \text{smallest value}$$
$$g(10) = 3(10)^2 - 12(10) - 2 = 178$$

so the point(s) on the curve $y = x^3 - 6x^2 \; 2x + 5$ with the tangent line having the smallest slope is

$$\boxed{(2, -15)}$$

3 (a) $\int \cos(x)(\sin(x))^{1/3}\,dx$

use substitution: Let $u = \sin(x)$

then $\frac{du}{dx} = \cos(x)$ so $dx = \frac{du}{\cos(x)}$

so $$\int \cos(x)(\sin(x))^{1/3}\,dx = \int \cos(x)(u)^{1/3}\frac{du}{\cos(x)}$$

$$= \int u^{1/3}\,du$$

$$= \frac{u^{1/3+1}}{1/3+1} + C$$

$$= \frac{3u^{4/3}}{4} + C$$

$$= \boxed{\frac{3(\sin(x))^{4/3}}{4} + C}$$

(b) $$\int (1-x^2)^2\,dx = \int (1-x^2)(1-x^2)\,dx$$

$$= \int (1 - x^2 - x^2 + x^4)\,dx$$

$$= \int (1 - 2x^2 + x^4)\,dx$$

$$= \boxed{x - \frac{2x^3}{3} + \frac{x^5}{5} + C}$$

(4) (a) $f(x) = 5x^2$

$$\Delta x = \frac{b-a}{n} = \frac{2-0}{n} = \frac{2}{n}$$

I will use the right endpoints as the sample points (x_i^*) so we have:

$$x_i = a + i\Delta x = 0 + i\left(\frac{2}{n}\right) = \frac{2i}{n}$$

hence

$$\int_0^2 5x^2\,dx = \boxed{\lim_{n\to\infty} \sum_{i=1}^{n} 5\left(\frac{2i}{n}\right)^2\left(\frac{2}{n}\right)}$$

(b) $$\int_0^3 x\,dx = \left[\frac{x^2}{2}\right]_0^3 = \frac{3^2}{2} - \frac{0^2}{2} = \boxed{\frac{9}{2}}$$

(c) use substitution:

Let $u = 1 + x^2$ $\Rightarrow$ so $u(0) = 1 + 0^2 = 1$

$u(\sqrt{8}) = 1 + (\sqrt{8})^2 = 1 + 8 = 9$

$$\frac{du}{dx} = 2x \Rightarrow \text{so } dx = \frac{du}{2x}$$

now $$\int_0^{\sqrt{8}} 3x\sqrt{1+x^2}\,dx = \int_1^9 3x\sqrt{u}\,\frac{du}{2x} = \frac{3}{2}\int_1^9 u^{1/2}\,du$$

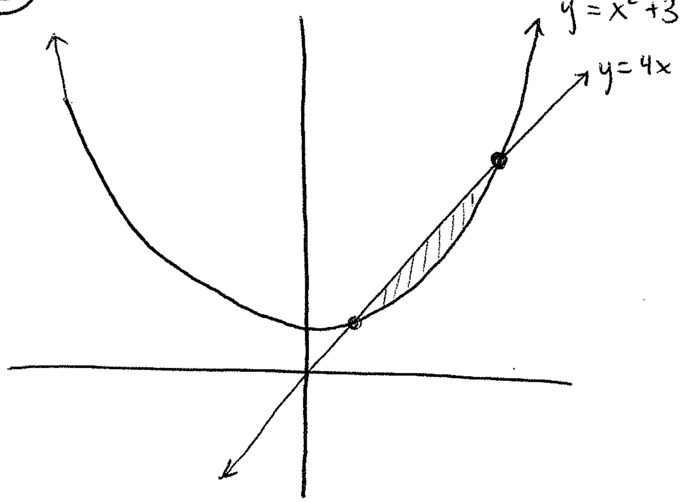
$$= \frac{3}{2}\left[\frac{u^{3/2}}{3/2}\right]_1^9$$

$$= 9^{3/2} - 1^{3/2}$$

$$= 27 - 1$$

$$= \boxed{26}$$

(5)



points of intersection:

$$x^2 + 3 = 4x$$
$$x^2 - 4x + 3 = 0$$
$$(x-3)(x-1) = 0$$
$$x = 3, \quad x = 1$$

which function is above the other over the interval $(1,3)$?

from the graph we see that $y = 4x$ is above $y = x^2+3$ over the interval $(1,3)$

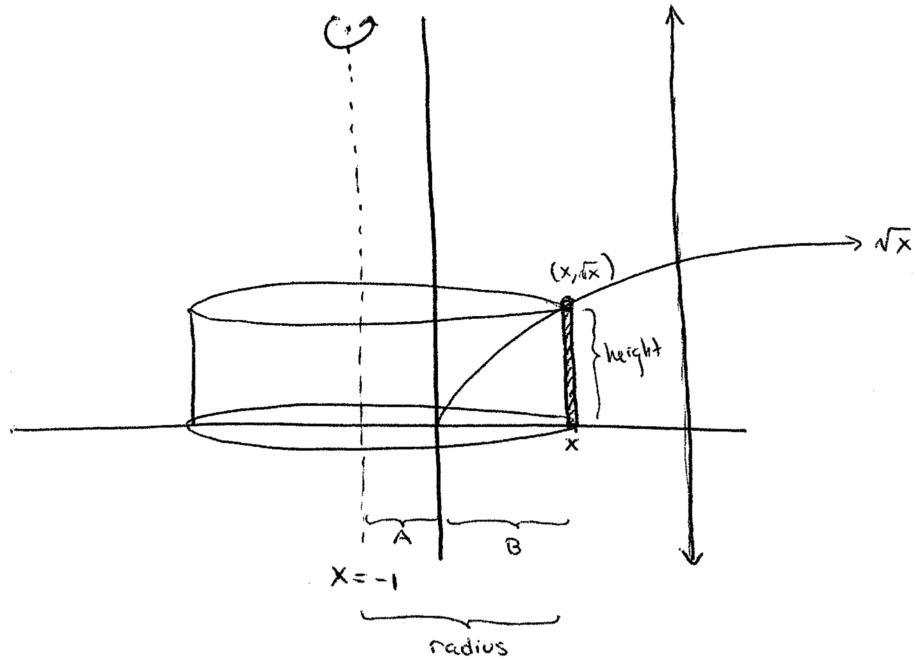
(you could also use a sign diagram)

so $$\boxed{A = \int_1^3 \left((4x) - (x^2+3)\right) dx}$$

(you could also solve this by integrating with respect to y but it is a bit harder)

(6)

solution 1: (shell method)



draw an area component parallel to the axis of rotation $(x = -1)$ as shown.

as x changes we integrate from $x = 0$ to $x = 4$

radius: $1 + x$

(notice the radius $= A + B$ and $A = 1$, $B = x$ hence radius $= 1 + x$)

height: $\sqrt{x}$

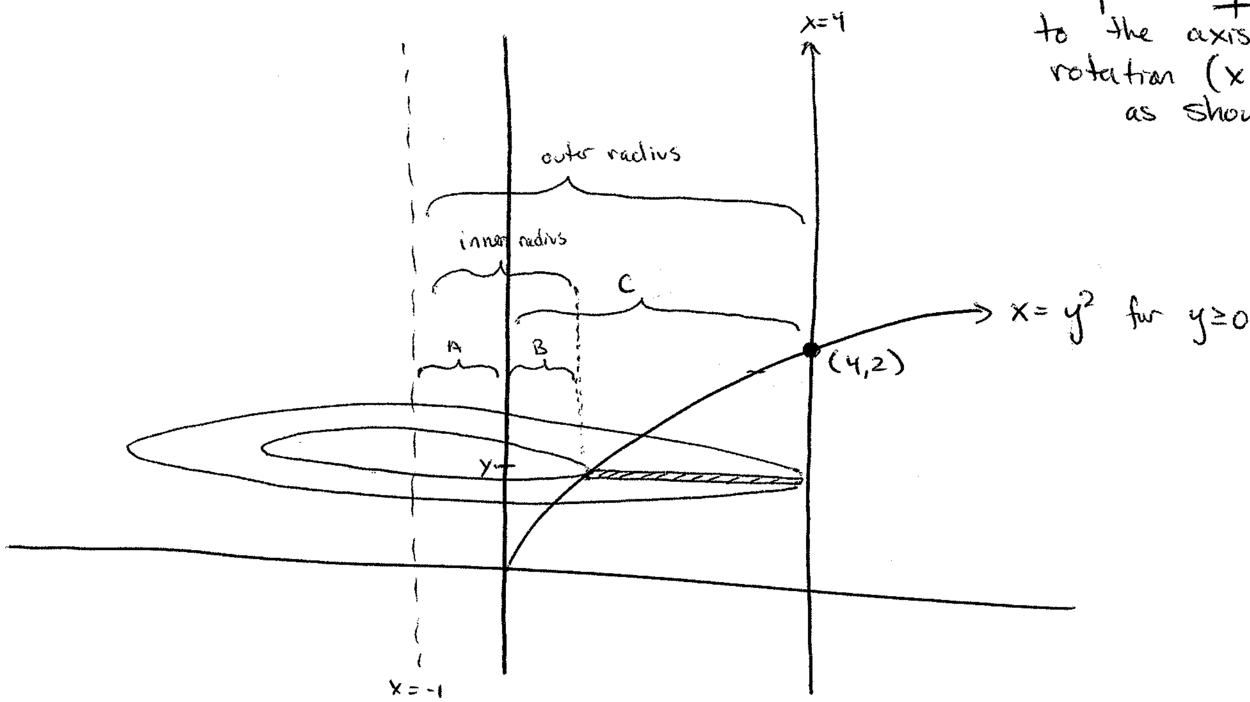
cross-sectional area: $2\pi(1+x)(\sqrt{x}) = A(x)$

hence

$$V = \int_a^b A(x)\,dx = \boxed{\int_0^4 2\pi(1+x)(\sqrt{x})\,dx}$$

solution 2 : (washer method)

draw an area component perpendicular to the axis of rotation ($x=-1$) as shown.



as y changes we integrate from $y=0$ to $y=2$
(this can be seen from the graph or intersection points $4=y^2 \iff y=\pm 2$ if $y\geq 0$ then $y=2$)

inner radius : $1+y^2$ (notice inner radius $= A+B = 1+y^2$)

outer radius : 5 (notice outer radius $= A+C = 1+4=5$)

cross-sectional area : $A(y) = \pi(5)^2 - \pi(1+y^2)^2$
$= \pi\left((5)^2-(1+y^2)^2\right)$

hence

$$V = \int_c^d A(y)\,dy = \boxed{\int_0^2 \pi\left((5)^2-(1+y^2)^2\right)dy}$$