Quiz #17 - Homework Quiz.

5.1 p251 #22

find two power series solutions of the given differential equation about the ordinary point $x=0$:

$$y'' + 2xy' + 2y = 0$$

SOL:

$$y = \sum_{n=0}^{\infty} C_n x^n$$

$$y' = \sum_{n=0}^{\infty} n C_n x^{n-1} = \sum_{n=1}^{\infty} n C_n x^{n-1}$$

$$y'' = \sum_{n=0}^{\infty} n(n-1) C_n x^{n-2} = \sum_{n=2}^{\infty} n(n-1) C_n x^{n-2}$$

so

$$y'' + 2xy' + 2y = 0 \quad \text{becomes.}$$

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} + 2x \sum_{n=1}^{\infty} n C_n x^{n-1} + 2 \sum_{n=0}^{\infty} C_n x^n = 0$$

$$\sum_{n=2}^{\infty} n(n-1) C_n x^{n-2} + \sum_{n=1}^{\infty} 2n C_n x^n + \sum_{n=0}^{\infty} 2C_n x^n = 0$$

starts at $x^0$ starts at $x^1$ starts at $x^0$

Now take out terms so the series start with the same power of x.

$$2(2-1) C_2 x^{2-2} + \sum_{n=3}^{\infty} n(n-1) C_n x^{n-2} + \sum_{n=1}^{\infty} 2n C_n x^n + 2C_0 x^0 + \sum_{n=1}^{\infty} 2C_n x^n = 0$$

starts at $x^1$ $(n=3)$ starts at $x^1$ $(n=1)$ start at $x^1$ $(n=1)$

Now shift so they series all start at the same index value. (I will shift to start at $n=1$)

$$2C_2 + 2C_0 + \sum_{n=1}^{\infty} (n+2)(n+1) C_{n+2} x^n + \sum_{n=1}^{\infty} 2n C_n x^n + \sum_{n=1}^{\infty} 2C_n x^n = 0$$

$$2C_2 + 2C_0 + \sum_{n=1}^{\infty} \left[ (n+2)(n+1) C_{n+2} x^n + 2n C_n x^n + 2C_n x^n \right] = 0$$

$$2C_2 + 2C_0 + \sum_{n=1}^{\infty} \left[ (n+2)(n+1) C_{n+2} + 2n C_n + 2C_n \right] x^n = 0$$

By the identity property

$$2C_2 + 2C_0 = 0 \quad \underline{\text{AND}} \quad (n+2)(n+1) C_{n+2} + 2n C_n + 2C_n = 0$$

for $n = 1, 2, 3, \ldots$

Always solve for higher coef.

$$C_2 = -C_0 \quad \underline{\text{AND}} \quad C_{n+2} = \frac{-2n C_n - 2C_n}{(n+2)(n+1)} \quad n = 1,2,3,\ldots$$

$$= \frac{(-2n-2) C_n}{(n+2)(n+1)} \quad n = 1,2,3,\ldots$$

$$= \frac{-2\cancel{(n+1)} C_n}{(n+2)\cancel{(n+1)}} \quad n = 1,2,3,\ldots$$

$$= \frac{-2}{n+2} C_n \quad , n = 1,2,3,\dots$$

set up the table:

$$C_0 = ?$$
$$C_1 = ?$$
$$C_2 = -C_0$$

| $n$ | $C_{n+2} = \frac{-2}{n+2} C_n$ |
|---|---|
| 1 | $C_3 = \frac{-2}{1+2} C_1 = \frac{-2}{3} C_1$ |
| 2 | $C_4 = \frac{-2}{2+2} C_2 = \frac{-2}{4} C_2 = -\frac{1}{2}(-C_0) = \frac{1}{2} C_0$ |
| 3 | $C_5 = \frac{-2}{3+2} C_3 = \frac{-2}{5} C_3 = \frac{-2}{5}\left(\frac{-2}{3}\right) C_1 = \frac{4}{15} C_1$ |
| 4 | $C_6 = \frac{-2}{4+2} C_4 = \frac{-2}{6} C_4 = \frac{-2}{6}\left(\frac{1}{2}\right) C_0 = -\frac{1}{6} C_0$ |
| 5 | $C_7 = \frac{-2}{5+2} C_5 = \frac{-2}{7} C_5 = \frac{-2}{7}\left(\frac{4C_1}{15}\right) = \frac{-8}{105} C_1$ |

$\vdots$

So all the way back at the beginning we had $y = \sum_{n=0}^{\infty} C_n x^n$

So $y = C_0 + C_1 x + C_2 x^2 + C_3 x^3 + \dots$

$$y = C_0 + C_1 x + (-C_0)x^2 + \left(-\frac{2}{3}C_1\right)x^3 + \left(\frac{1}{2}C_0\right)x^4 + \left(\frac{4C_1}{15}\right)x^5 + \left(-\frac{1}{6}C_0\right)x^6 + \left(-\frac{8}{105}C_1\right)x^7 + \dots$$
$$= \left(C_0 - C_0 x^2 + \frac{1}{2}C_0 x^4 - \frac{1}{6}C_0 x^6 + \dots\right) + \left(C_1 x - \frac{2}{3}C_1 x^3 + \frac{4}{15}C_1 x^5 - \frac{8}{105}C_1 x^7 + \dots\right)$$
$$= C_0 \underbrace{\left(1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \dots\right)}_{y_1} + C_1 \underbrace{\left(x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{105}x^7 + \dots\right)}_{y_2}$$

so

$$\boxed{y_1 = 1 - x^2 + \frac{1}{2}x^4 - \frac{1}{6}x^6 + \dots}$$

AND

$$\boxed{y_2 = x - \frac{2}{3}x^3 + \frac{4}{15}x^5 - \frac{8}{105}x^7 + \dots}$$