Quiz # 9 – HW Quiz

3.5 p132 (#12)

Solve $y'' - 2y' + y = \dfrac{e^x}{1+x^2}$ by variation of parameters.

SOLUTION :

Step 1: Find y_c of $y'' - 2y' + y = 0$

the auxiliary equation is :

$$m^2 - 2m + 1 = 0$$
$$(m-1)(m-1) = 0$$
$$m = 1, \ m = 1, \quad \leftarrow \text{repeated roots (case II on p 113)}.$$

so $$y_c = \underbrace{c_1 e^x}_{y_1} + \underbrace{c_2 x e^x}_{y_2}$$

Step 2: Find y_p using variation of parameters.

$$y_1 = e^x$$
$$y_2 = x e^x$$

$$W = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = \begin{vmatrix} e^x & xe^x \\ e^x & xe^x + e^x \end{vmatrix} = e^x(xe^x + e^x) - xe^x e^x$$
$$= xe^{2x} + e^{2x} - xe^{2x}$$
$$= e^{2x}$$

to find $f(x)$, put the D.E. into standar form. (Already DONE!)

~~So f(x)~~ Standard Form $y'' - 2y' + y = \underbrace{\dfrac{e^x}{1+x^2}}_{f(x)}$

So $$f(x) = \frac{e^x}{1+x^2}$$

Now $$u_1' = \frac{-y_2 f(x)}{W} = \frac{-xe^x \cdot \left(\frac{e^x}{1+x^2}\right)}{e^{2x}} = \frac{-x\cancel{e^{2x}}}{\cancel{e^{2x}}(1+x^2)}$$
$$= \frac{-x}{1+x^2}$$

so $$u_1 = \int \frac{-x}{1+x^2}\,dx$$

Let $t = 1+x^2 \Rightarrow \dfrac{dt}{dx} = 2x$

$\Rightarrow dx = \dfrac{dt}{2x}$

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so $$u_1 = \int \frac{-x}{1+x^2}\,dx = \int \frac{-x}{t}\cdot\frac{dt}{2x}$$

$$= -\frac{1}{2}\int \frac{1}{t}\,dt$$

$$= -\frac{1}{2}\ln|t|$$

$$= -\frac{1}{2}\ln|1+x^2|$$

you can actually get rid of the absolute values since $1+x^2 \geq 0$ for all $x \in \mathbb{R}$.

$$= -\frac{1}{2}\ln(1+x^2)$$

and $$u_2' = \frac{y_1 f(x)}{W} = \frac{e^x \cdot \left(\frac{e^x}{1+x^2}\right)}{e^{2x}} = \frac{\cancel{e^{2x}}}{\cancel{e^{2x}}(1+x^2)} = \frac{1}{1+x^2}$$

so $$u_2 = \int \frac{1}{1+x^2}\,dx = \tan^{-1}(x)$$

hence $$y_p = u_1 y_1 + u_2 y_2 = \left(-\frac{1}{2}\ln(1+x^2)\right)(e^x) + \left(\tan^{-1}(x)\right)xe^x$$

Step 3: the general solution is

$$y = \underbrace{y_c}_{\text{step 1}} + \underbrace{y_p}_{\text{step 2}}$$

$$y = c_1 e^x + c_2 x e^x + \left(-\frac{1}{2}\ln(1+x^2)\right)e^x + \tan^{-1}(x)\,xe^x$$

$$\boxed{y = c_1 e^x + c_2 x e^x - \frac{1}{2}e^x \ln(1+x^2) + xe^x \tan^{-1}(x)}$$